Integrating Powers of Quadratics

Only Example 7.4.8 in Stewart (4th edition) requires integrating a rational function with a repeated quadratic in the denominator. That example does not deal with the full generality of the problem. For example, after reading Section 7.4 you will still not be able to evaluate $\int \frac{dx}{(x^2+1)^2}$. Normally, this would not be a problem since a table of integrals would help. Unfortunately, the tables in Stewart lack the standard section called "Forms Involving $(ax^2+bx+c)$". Here are two entries from a typical table:

$$\int \frac{dx}{(ax^2+bx+c)^{n+1}} = \frac{2ax+b}{nq(ax^2+bx+c)^n} + \frac{2(2n-1)a}{nq}\int \frac{dx}{(ax^2+bx+c)^n} \tag{1}$$

$$\int \frac{x\,dx}{(ax^2+bx+c)^{n+1}} = \frac{-(2c+bx)}{nq(ax^2+bx+c)^n} - \frac{(2n-1)b}{nq}\int \frac{dx}{(ax^2+bx+c)^n} \tag{2}$$

where $n > 0$ and $q = 4ac - b^2$. You can verify them by differentiating both sides. The formulas are valid even if the quadratic factors into a product of two linear factors. You can also derive them in a couple of ways.

The first approach is through a somewhat complicated integration by parts. Call the two integrals $I_0(n+1)$ and $I_1(n+1)$ and let $Y = ax^2+bx+c$. The table entries can then be written more concisely as

$$I_0(n+1) = \frac{2ax+b}{nqY^n} + \frac{2(2n-1)aI_0(n)}{qn} \quad \text{and} \quad I_1(n+1) = \frac{-(2c+bx)}{nqY^n} - \frac{(2n-1)bI_0(n)}{qn}.$$

First, use integration by parts on $I_0(n)$ with $u = Y^n$ and $dv = dx$ to obtain

$$\begin{aligned} I_0(n) &= \frac{x}{Y^n} + \int \frac{nx(2ax+b)\,dx}{Y^{n+1}} = \frac{x}{Y^n} + n\int \frac{2(ax^2+bx+c) - bx - 2c}{Y^{n+1}}\,dx \\ &= \frac{x}{Y^n} + \frac{2n}{Y^n} - bnI_1(n+1) - 2cnI_0(n+1). \end{aligned}$$

Rearranging, we obtain

$$bnI_1(n+1) + 2cnI_0(n+1) = \frac{x}{Y^n} + (2n-1)I_0(n) \tag{3}$$

Second, note that

$$2aI_1(n+1) + bI_0(n) = -1/nY^n \tag{4}$$

by using the substitution $t = ax^2 = bx + c$. Equations (3) and (4) can be solved for the two unknowns $I_0(n+1)$ and $I_1(n+1)$.

The second approach is by a trig substitution, provided the quadratic does not factor. In that case, one can complete the square to obtain

$$ax^2 + bx + c = a\Big((x + b/2a)^2 + r^2\Big) \quad \text{where} \quad r = \frac{\sqrt{4ac - b^2}}{2a}$$

and $r$ is real. The substitution $x + b/2 = r \tan t$ converts (1) into

$$\int \frac{\sec^2 t \; dt}{(ar^2)^{n+1} \sec^{2n+2} t} = (4a/q)^{n+1} \int \cos^{2n} t \; dt, \tag{5}$$

which can be done by the methods in Section 7.2. However, to obtain (1), it's necessary to write

$$\int \cos^{2n} t \; dt = \int \cos^{2n-2} t \; dt - \int \cos^{2n-2} t \; \sin^2 t \; dt \tag{6}$$

and use integration by parts on the last integral with $u = \sin t$ and $dv = \cos^{2n-2} t \; \cos t \; dt$ from which we have $v = (-\cos 2n - 1t)/(2n-1)$. After some rearranging and substituting back to eliminate $t$. In the process, you must remember that (5) is $I_0(n+1)$ so that the middle integral in (6) differs from $I_0(n)$ by factor of $(4a/q)^n$. The details are left to the reader.

**Example** We compute $\int \frac{dx}{(x^2+4)^3}$. Apply (1) twice, first with $n = 2$ and then with $n = 1$:

$$\begin{aligned}
\int \frac{dx}{(x^2+4)^3} &= \frac{8x}{32(4x^2+1)^2} + \frac{24}{32} \int \frac{dx}{(4x^2+1)^2} \\
&= \frac{x}{4(4x^2+1)^2} + \frac{3}{4} \left( \frac{8x}{16(4x^2+1)} + \frac{8}{16} \int \frac{dx}{4x^2+1} \right) \\
&= \frac{x}{4(4x^2+1)^2} + \frac{3x}{8(x^2+1)} + \frac{3}{8} \frac{\arctan 2x}{2} + C,
\end{aligned}$$

where the last integral was done by pulling out a factor of 4 from the denominator.