Section 7.4 An Easier Method for Finding Some Partial Fractions

We'll explore a different way to get the parts of the partial fraction expansion whose denominators are powers of a linear factor. This method often involves less work since there are no equations to solve.

Suppose we want to expand $p(x)/q(x)$ where the degree of $p(x)$ is less than the degree of $q(x)$ and that $p(x)$ and $q(x)$ have no common factor. Suppose also that $q(x) = x^n r(x)$ where $r(x)$ is not divisible by x. Imagine putting all the partial fractions that have a power of x in the denominator over a common denominator. Imagine doing the same for all the remaining partial fractions. When this is done, we will have the equation

$$\frac{p(x)}{x^n\, r(x)} = \frac{A(x)}{x^n} + \frac{B(x)}{r(x)}, \tag{1}$$

where the degree of $A(x)$ is less than n. (We're just asserting that this is true, not proving it.)

If we know the coefficients in $A(x) = a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$, we can easily divide by x^n to obtain the part of partial fraction for powers of x:

$$\frac{A(x)}{x^n} = \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_{n-1}}{x^{n-1}} + \frac{a_n}{x^n}.$$

How do we find $A(x)$?

We can find $A(x)$ rather easily. From (1) we have

$$\frac{p(x)}{x^n\, r(x)} = \frac{r(x)A(x) + x^n\, B(x)}{x^n r(x)}$$

and so

$$p(x) = r(x)A(x) + x^n B(x), \tag{2}$$

where the coefficients of $A(x)$ and $B(x)$ are unknown. Equate the coefficients of x^k in (2) for $0 \le k < n$. Notice that the coefficients of $B(x)$ don't appear since $B(x)$ is multiplied by x^n. Thus we have

$$\begin{array}{lll}
\text{coef. of } x^0\text{:} & p_0 = r_0a_0 & \text{whence} \quad a_0 = p_0/r_0 \\
\text{coef. of } x^1\text{:} & p_1 = r_0a_1 + r_1a_0 & \text{whence} \quad a_1 = (p_1 - r_1a_0)/r_0 \\
\text{coef. of } x^2\text{:} & p_2 = r_0a_2 + r_1a_1 + r_2a_0 & \text{whence} \quad a_2 = (p_2 - r_1a_1 - r_2a_0)/r_0 \\
\quad \cdots & & \\
\text{coef. of } x^k\text{:} & p_k = r_0a_k + \cdots + r_ka_0 & \text{whence} \quad r_k = (p_k - r_1a_{k-1} - \cdots - r_ka_0)/r_0 \\
\quad \cdots & &
\end{array}$$

Use the first equation to get a_0, then use the second to get a_1 and so forth.

This sounds more complicated than it is, so let's work an example.

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Example Consider

$$\frac{x^2+5x-2}{x^4(x+1)^3}. \tag{3}$$

We write this as

$$\frac{x^2+5x-2}{x^4(x+1)^3} = \frac{A(x)}{x^4} + \frac{B(x)}{(x+1)^3}.$$

Since $r(x) = (x+1)^3 = x^3 + 3x^2 + 3x + 1$, we have

$$x^2 + 5x - 2 = (x^3 + 3x^2 + 3x + 1)A(x) + x^4 B(x).$$

Thus

$$\begin{array}{lrl} \text{coef. of } x^0\text{:} & -2 = 1a_0 & \text{and so } a_0 = -2 \\ \text{coef. of } x^1\text{:} & 5 = 1a_1 + 3a_0 & \text{and so } a_1 = 11 \\ \text{coef. of } x^2\text{:} & 1 = 1a_2 + 3a_1 + 3a_0 & \text{and so } a_2 = -26 \\ \text{coef. of } x^3\text{:} & 0 = 1a_3 + 3a_2 + 3a_1 + 1a_0 & \text{and so } a_3 = 47 \end{array}$$

We have part of the partial fraction expansion: $-2/x^4 + 11/x^3 - 26/x^2 + 47/x$.

If this only worked for powers of x in the denominator, it wouldn't be much use. With a change of variable, we can make it work for any linear factor: Suppose we want partial fractions for $p(x)/q(x)$ and we have $q(x) = (cx+d)^m s(x)$.

- Change variables by setting $y = cx + d$; that is, $x = (y-d)/c$.
- Apply the preceding method to obtain $a_1/y + \cdots + a_m/y^m$.
- Now replace y with $cx + d$ to get the partial fraction we want!
- Repeat this process with all the distinct linear factors.

Example We continue with (3) of the previous example to get the complete partial fraction expansion. Set $y = x + 1$ so $x = y - 1$. Then

$$x^2 + 5x - 2 = (y-1)^2 + 5(y-1) - 2 = y^2 + 3y - 6.$$

Since

$$x^4 = (y-1)^4 = y^4 - 4y^3 + 6y^2 - 4y + 1,$$

the equations and there solutions are

$$\begin{array}{lrl} \text{coef. of } y^0\text{:} & -6 = 1a_0 & \text{and so } a_0 = -6 \\ \text{coef. of } y^1\text{:} & 3 = 1a_1 - 4a_0 & \text{and so } a_1 = -21 \\ \text{coef. of } y^2\text{:} & 1 = 1a_2 - 4a_1 + 6a_0 & \text{and so } a_2 = -47 \end{array}$$

Thus

$$\frac{x^2+5x-2}{x^4(x+1)^3} = -\frac{2}{x^4} + \frac{11}{x^3} - \frac{26}{x^2} + \frac{47}{x} - \frac{6}{(x+1)^3} - \frac{21}{(x+1)^2} - \frac{47}{x+1}.$$