Please start each problem on a NEW PAGE.

Remember to show your work.

1. (25 pts.) Define the following:
   (a) an eigenvalue of the $n \times n$ matrix A,
   (b) the null space of an $m \times n$ matrix A,
   (c) the dimension of a vector space V.

2. (15 pts.) Show that $T(x_1, x_2, x_3) = (3x_2 - x_3, x_1 - 4x_2, x_3)$ is a linear transformation and find its standard matrix.

3. (25 pts.) Find the eigenvalues and eigenspaces of $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & -1 \end{bmatrix}$.

4. (20 pts.) Let W be the span of the orthogonal vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$. Write $\begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$ as the sum of a vector in the subspace W and a vector orthogonal to W.

5. (20 pts.) Find the eigenvalues associated with the quadratic form $8x_1^2 + 6x_1x_2$ and use them to classify the form.

6. (60 pts.) Suppose that A is an $n \times n$ matrix and that R is the reduced row echelon form of A. You are given R but *are not given* A. For each of the following explain why you can or cannot answer it given R but not A.
   (a) Does A^{-1} exist?
   (b) What is a basis for $\operatorname{Nul} A$?
   (c) What is a basis for $\operatorname{Col} A$?
   (d) What is a basis for $\operatorname{Row} A$?
   (e) What are the eigenvalues of A?
   (f) Does $A\mathbf{x} = \mathbf{b}$ have a solution when $\mathbf{b}$ is the first column of the identity matrix I?

7. (20 pts.) A scientist solves a nonhomogeneous system of seven (7) linear equations in ten (10) unknowns and finds that the general solution has three (3) free parameters. Can the scientist be certain that, if the right sides of the equations are changed, the new nonhomogeneous system will have a solution? *You must justify your answer to receive credit.*

8. (20 pts.) Prove that, if
   (a) A is an $n \times n$ matrix with n linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and
   (b) $P = [\mathbf{v}_1 \, \mathbf{v}_2 \ldots \mathbf{v}_n]$,

   then the *rows* of P^{-1} are eigenvectors of A^T.

9. (20 pts.) Suppose A and B are $n \times n$ matrices and AB is invertible. Prove that B is invertible.
   Hint: Consider $AB\mathbf{x} = \mathbf{0}$.

End of exam. Have a good summer.