1. (22 pts.) (a) Define a basis $\mathcal{B}$ of a vector space V.

   (b) Fill in the following definition: Let A be an $n \times n$ matrix. If there is a matrix C such that ?????, then we call C the inverse of A.

2. (36 pts.) Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -2 & -2 \\ 2 & 1 & 1 & 4 & 5 \\ 1 & 0 & 0 & 3 & 3 \end{bmatrix}$. I found that $\begin{bmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is the reduced echelon form of A. Give bases and dimensions for the following three spaces.

$$\operatorname{Col} A \qquad \operatorname{Nul} A \qquad \operatorname{Row} A.$$

3. (40 pts.) For each of the following answer either "Yes," "No," or "Not enough information."

To receive credit *you must justify your answers.*

You needn't quote theorems; for example, you could say things like "$(A^T)^{-1} = (A^{-1})^T$" or "$\dim H \le \dim V$ since H is a subspace of V." Also, for "not enough information" you could give two examples—one where the statement is true and another where it is false.

   (a) A is a 6×6 matrix of real numbers and the reduced echelon form for A is I_6, the 6×6 identity matrix. Does the determinant of A equal -4?

   (b) A is the same matrix as in (a). Does the inverse of A exist?

   (c) A is the same matrix as in (a) and (b).
   Is the linear transformation from $\mathbb{R}^6$ to $\mathbb{R}^6$ given by $\vec{x} \mapsto A\vec{x}$ one-to-one?

   (d) B is a 6×8 matrix. The solutions to $B\vec{x} = \vec{0}$ form a subspace of dimension 3.
   Is it possible to find a vector $\vec{b}$ such that $B\vec{x} = \vec{b}$ is inconsistent?

4. (27 pts.) Suppose that A and B are matrices whose dimensions are such that AB is defined. The goal of this problem is to prove the inequality

$$\operatorname{rank}(AB) \le \min(\operatorname{rank} A,\ \operatorname{rank} B).$$

Note that you can use the result (a) in proving (b), even if you have not done (a). Similarly, you can use (b) in proving (c), even if you have not done (a) or (b).

   (a) Prove that the columns of AB are in $\operatorname{Col} A$.
   Hint: The jth column of AB is $A\vec{b}_j$ where $\vec{b}_j$ is the jth column of B.

   (b) Using (a) or otherwise, prove that $\operatorname{rank}(AB) \le \operatorname{rank} A$.
   Hint: Show that $\operatorname{Col}(AB)$ is a subspace of $\operatorname{Col} A$.

   (c) Using Theorem 5.15, it can easily be shown that $\operatorname{rank} D^T = \operatorname{rank} D$ for every matrix, but you do not need to prove it. Using (b) and the fact that $\operatorname{rank} D^T = \operatorname{rank} D$ prove that $\operatorname{rank}(AB) \le \operatorname{rank} B$. *Hint*: Remember that $(AB)^T = B^T A^T$.

(Exam #1 had 100 points for 20% of the grade. This exam has 125 points for 25%.)