1. See pages 42 and 74, respectively.

2. Use row reduction of A augmented by $\vec{b}$. To make work easier, I started with the fourth column.

$$\begin{bmatrix}1&0&-1&0&1\\0&0&0&1&0\\0&1&1&-1&0\\1&2&1&3&0\end{bmatrix}\sim\begin{bmatrix}1&0&-1&0&1\\0&0&0&1&0\\0&1&1&0&0\\1&2&1&0&0\end{bmatrix}\sim\begin{bmatrix}1&0&-1&0&1\\0&0&0&1&0\\0&1&1&0&0\\0&2&2&0&-1\end{bmatrix}$$

$$\sim\begin{bmatrix}1&0&-1&0&1\\0&0&0&1&0\\0&1&1&0&0\\0&0&0&0&-1\end{bmatrix}\sim\begin{bmatrix}1&0&-1&0&1\\0&1&1&0&0\\0&0&0&1&0\\0&0&0&0&-1\end{bmatrix}$$

You may have taken a different route, but should get the same answer since reduced row echelon form is unique.

(a) For $A\vec{x}=\vec{0}$, the last column in the above reduction would be replaced by all zeroes. Since the pivot columns are 1, 2 and 4, it follows that x_3 is free. From the reduced form, $x_1=x_3$, $x_2=-x_3$ and $x_4=0$. That's one description. Another is $\vec{x}=x_3\begin{bmatrix}1\\-1\\1\\0\end{bmatrix}$.

(b) From above reduction, $A\vec{x}=\vec{b}$ is inconsistent and so there are no solutions.

3. (a) No. Linear independence means $A\vec{x}=\vec{0}$ has only the trivial solution. From M we can see that x_4 is free, so there are nontrivial solutions.

(b) Yes. See (a).

(c) Not enough information. By (b) we know that when $\vec{b}=\vec{0}$ there is more than one solution. If we augment A with $\vec{b}$ and carry out reduction, it may happen that the lower right corner will be nonzero. In this case, the equations will be inconsistent and so there will be no solutions. Problem 1 provides an example of how this can happen.

4. Let A be the standard matrix for T. Here are some ways the proof could be done.

• *one-to-one*: By the definition, $A\vec{x}=\vec{b}$ has at most one solution for each $\vec{b}$. This means that there are no free varialbes and so each column of A is a pivot column. Thus there are n pivots and so at least n rows. Hence $n\le m$.

• *one-to-one* (alternate): Theorem 9 (p.83) tells us that $A\vec{x}=\vec{0}$ has only the trivial solution. This means that there are no free variables, so proceed as before.

• *one-to-one* (alternate): Theorem 10(b) (p.83) tells us that the columns of A are linearly independent. The contrapositive of Theorem 6 (p.67) (with $p=n$ and $\mathbb{R}^n=\mathbb{R}^m$) tells us that $n\le m$.

• *onto*: The definition tells us that $A\vec{x}=\vec{b}$ has a solution for all $\vec{b}$. Theorem 2 (p.51) tells us that A has a pivot in every row. Hence there are m pivots. Since each column has at most one pivot $n\ge m$.

• *onto* (alternate): Theorem 10(a) (p.83) tells us that the columns of A span $\mathbb{R}^m$. Again, apply Theorem 2 (p.51).