1. Since the matrix is triangular, the eigenvalues are the diagonal entries. Finding eigenvectors involves finding bases for the null spaces of $A - \lambda I$. The answers are

$$\lambda = 1 \quad \mathbf{v} = \alpha(1,0,0)^T \qquad \lambda = 3 \quad \mathbf{v} = \beta(1,1,0)^T \qquad \lambda = 4 \quad \mathbf{v} = \gamma(0,0,1)^T$$

where α, β and γ are nonzero scalars. Any values you chose for those scalars are correct.

2. (a) The easiest way to construct an orthonormal basis is to note that the vectors are orthogonal and each has length 3. This gives us

$$\begin{aligned} \mathbf{u}_1 &= (1/3,\ 2/3,\ 0,\ 2/3,\ 0)^T \\ \mathbf{u}_2 &= (2/3,\ -1/3,\ 0,\ 0,\ 2/3)^T \\ \mathbf{u}_3 &= (0,\ 2/3,\ 0,\ -2/3,\ 1/3)^T. \end{aligned}$$

You could use Gram-Schmidt orthogonalization and obtain the same answer. The orthonormal basis is not unique, so you may have obtained a different correct answer; however, that's fairly unlikely since this choice is the obvious one.

(b) If the given vector is $\mathbf{v}$, then $\mathbf{v}^T\mathbf{u}_1 = 15$, $\mathbf{v}^T\mathbf{u}_2 = 9$ and $\mathbf{v}^T\mathbf{u}_3 = 3$. Thus the projection of $\mathbf{v}$ onto W is $15\mathbf{u}_1 + 9\mathbf{u}_2 + 3\mathbf{u}_3 = \mathbf{p} = (11,\ 9,\ 0,\ 8,\ 7)^T$. Thus $\mathbf{q} = \mathbf{v} - \mathbf{p} = (-2,\ 0,\ 9,\ 1,\ 2)$ is in $W^\perp$ and so $\mathbf{v} = \mathbf{p} + \mathbf{q}$ is the answer. (There is no other correct answer.)

3. $L = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

4. Recall that the determinant is the product of the eigenvalues and so $\det(A) = 6$.
   (i) 1, -1, 1/2 and 1/3; $\det(A^{-1}) = -1/6$
   (ii) Same as A.
   (iii) If $A\mathbf{v} = \lambda\mathbf{v}$, then $p(A)\mathbf{v} = p(\lambda)\mathbf{v}$. Thus we have 0, 2, 2 and 6; $\det(A^2 - A) = 0$.

5. Using the hint, column k of A is $w_k\mathbf{v}$. Hence all the columns are multiples of $\mathbf{v}$ and so lie in the space spanned by $\mathbf{v}$. We need to show that $\mathbf{v}$ is in the span of the columns. Since we are given $\mathbf{w} \neq \mathbf{0}$, there is some j with $w_j \neq 0$. Multiplying column j by the scalar $1/w_j$, we obtain $\mathbf{v}$.

6. The only information about the row space $R(M^T)$ and the null space $N(M)$ of an arbitrary matrix $M \in \mathbb{R}^{n\times k}$ is:
   (i) $\dim(R(M)) \leq n$, the number of rows of M.
   (ii) Since $R(M^T) = N(M)^\perp$, $\dim(R(M^T)) + \dim(N(M)) = k$, the number of columns of M.

   A: Using the previous observations, we cannot rule this answer out In fact, there is a matrix with the given row and null space: Construct a matrix with that row space and the null space will automatically take care of itself.
   B: Since (0, 0′ 0, 1) and $(0,\ 1,\ 1,\ 2)^T$ are not orthogonal, the answer must be wrong.
   C: The spaces are orthogonal, but the sum of their dimensions is wrong, so C is incorrect. (The student must have missed a vector in the basis of either the row space or the null space.)

7. (i) Let $\mathbf{y} = A\mathbf{x}$. Recall that $\mathbf{y}^T\mathbf{y} \geq 0$ with equality if and only if $\mathbf{y} = \mathbf{0}$. Now

$$\langle \mathbf{x}, \mathbf{x}\rangle = (A\mathbf{x})^T(A\mathbf{x}) = \mathbf{y}^T\mathbf{y} \geq 0,$$

   with equality if and only if $\mathbf{y} = \mathbf{0}$. $\mathbf{y} = A\mathbf{x} = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$ because A is nonsingular.

   (ii) $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^T A^T A\mathbf{y} = (\mathbf{y}^T A^T A\mathbf{x})^T = (\langle \mathbf{y}, \mathbf{x}\rangle)^T$. Since the inner product is a scalar, it equals its transpose.

   (iii) $\langle \alpha\mathbf{x} + \beta\mathbf{y}, \mathbf{z}\rangle \;=\; (\alpha\mathbf{x} + \beta\mathbf{y})^T A^T A\mathbf{z} \;=\; \alpha\mathbf{x}^T A^T A\mathbf{z} + \beta\mathbf{y}^T A^T A\mathbf{z}$
   $= \alpha\langle \mathbf{x}, \mathbf{z}\rangle + \beta\langle \mathbf{y}, \mathbf{z}\rangle.$

8. It doesn't make sense to write $\det(A)$ because A is not a square matrix.