- There are a total of 80 points possible.
- TWO PAGES of notes are allowed. No calculators are allowed.
- **You must show your work to receive credit.**

1. (12 pts) (a) Find the eigenvalues of the matrix $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

   (b) Find an eigenvector for each eigenvalue.

2. (12 pts) Let W be the subspace of $\mathbb{R}^5$ spanned by $\begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{pmatrix}$.

   (a) Find an orthonormal basis for W.

   (b) Write $(9,\ 9\ 9,\ 9,\ 9)^T$ as a sum of a vector in W and a vector in $W^\perp$.

3. (6 pts) $L((a,b,c)^T) = ax(x-1) + bx + c$ defines a linear transformation L from $\mathbb{R}^3$ to P_3. Find a matrix for L using the standard basis for $\mathbb{R}^3$ and the basis 1, x, x^2 for P_3.

4. (12 pts) A matrix $A \in \mathbb{R}^{4\times 4}$ has eigenvalues 1, -1, 2 and 3. What are the eigenvalues and determinants of the following matrices?

   (i) A^{-1} (ii) A^T (iii) $A^2 - A$.

5. (10 pts) Let $\mathbf{v}$ and $\mathbf{w}$ be nonzero vectors in $\mathbb{R}^n$. Define $A \in \mathbb{R}^{n\times n}$ by $A = \mathbf{v}\mathbf{w}^T$. Prove that $\mathbf{v}$ is a basis for the column space of A.
   Hint: What is column k of A in terms of $\mathbf{v}$ and $\mathbf{w}$?

6. (10 pts) Given three matrices $A, B, C \in \mathbb{R}^{3\times 4}$ a student was told to compute a basis for the row space of each matrix and a basis for the null space of each matrix. The following answers were turned in.

	Row Space	Null Space
matrix A:	$(1, 0, \ \ 0, 0)$ $(0, 1, -1, 0)$	$\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$ $\begin{pmatrix}0\\1\\1\\2\end{pmatrix}$
matrix B:	$(1, 0, \ \ 0, 0)$ $(0, 1, -1, 0)$ $(0, 0, \ \ 0, 1)$	$\begin{pmatrix}0\\1\\1\\2\end{pmatrix}$
matrix C:	$(1, 0, \ \ 0, 0)$ $(0, 1, -1, 0)$	$\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$

The grader marked two answers wrong and one answer correct. It is possible to tell which answers are wrong without even knowing the matrices A, B, C!

Question: Which two answers must be wrong and why?

7. (12 pts) Suppose $A \in \mathbb{R}^{n\times n}$ is nonsingular. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, define $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T A^T A \mathbf{y}$. Prove that this makes $\mathbb{R}^n$ into an inner product space. That is, verify the three conditions in the definition of an inner product space:

   (i) $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ with equality if and only if $\mathbf{x} = \mathbf{0}$.

   (ii) $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$.

   (iii) $\langle \alpha\mathbf{x} + \beta\mathbf{y}, \mathbf{z} \rangle = \alpha\langle \mathbf{x}, \mathbf{z} \rangle + \beta\langle \mathbf{y}, \mathbf{z} \rangle$ for all $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ in $\mathbb{R}^n$ and all scalars α and β.

8. (6 pts) Let $A = (1, 0)$. It is easily seen that

$$\det(A^T A) = \det \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0 \quad \text{and} \quad \det(AA^T) = \det(1) = 1.$$

What is wrong with the following proof that $0 = 1$?

$$0 = \det(A^T A) = \det(A^T)\det(A) = \det(A)\det(A^T) = \det(AA^T) = 1.$$

END OF EXAM