Math 20E Final Exam 96 points 11:30AM Friday June 11, 2004

- Print Name, ID number and Section on your blue book.
- BOOKS and CALCULATORS are NOT allowed.
  Both sides of one page of NOTES is allowed.
- **You must show your work to receive credit.**

1. (12 points) Given that $\mathbf{A} \times \mathbf{B} \neq \mathbf{0}$, explain why the three vectors

$$\frac{\mathbf{A}}{|\mathbf{A}|}, \quad \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|} \quad \text{and} \quad \frac{(\mathbf{A} \times \mathbf{B}) \times \mathbf{A}}{|\mathbf{A} \times \mathbf{B}| \ |\mathbf{A}|}$$

are mutually orthogonal (i.e. mutually perpendicular) unit vectors.

2. (12 points) A surface is given by

$$\mathbf{R} \ = \ u \cos v \ \mathbf{i} \ - \ u \sin v \ \mathbf{j} \ + \ (4 - u^2)\mathbf{k} \quad \text{for} \ \ 0 \leq u \leq 2 \ \ \text{and} \ \ |v| \leq \pi.$$

(Note the minus sign on the $\mathbf{j}$ component.) Compute a unit normal to the surface such that the $\mathbf{k}$ component is positive.

3. (12 points) Find the value of the line integral

$$\int_C \big[(3x + 4y)dx + (2x + 3y^2)dy\big]$$

where C is the circle $x^2 + y^2 = 4$ traversed counterclockwise, that is, in the usual direction.

4. (12 points) Suppose that $\nabla \times \mathbf{H} = \mathbf{F} \times \mathbf{R}$. Prove that the curl of $\mathbf{F}$ is perpendicular to $\mathbf{R}$; that is, their dot product is zero.

Some identities: $\nabla \times (\mathbf{F} \times \mathbf{G}) \ = \ (\mathbf{G} \cdot \nabla)\mathbf{F} + (\nabla \cdot \mathbf{G})\mathbf{F} - \Big((\mathbf{F} \cdot \nabla)\mathbf{G} + (\nabla \cdot \mathbf{F})\mathbf{G}\Big)$,
$\nabla \cdot (\mathbf{F} \times \mathbf{G}) \ = \ \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$, $\nabla(x^2 + y^2 + z^2)/2 \ = \ \mathbf{R}$.

5. (12 points) When S is a surface with no boundary, Stokes' Theorem becomes

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = 0.$$

Use the Divergence Theorem to prove Stokes'Theorem in this case.

THERE ARE MORE PROBLEMS

6. (12 points) Suppose that $\mathbf{F}(\mathbf{R})$ is defined for all $\mathbf{R}$ and that $\nabla \cdot \mathbf{F}(\mathbf{R}) = 0$ for all $\mathbf{R}$. Let S be the portion of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the xy-plane and let $\mathbf{n}$ be the unit normal pointing outward from the sphere. Prove that

$$\iint_S \mathbf{F}(\mathbf{R}) \cdot \mathbf{n}\, dS \;=\; \int_0^{2\pi} \int_0^2 r F_3(r\cos\theta, r\sin\theta, 0)\, dr\, d\theta,$$

where $\mathbf{F} = F_1(x,y,z)\mathbf{i} + F_2(x,y,z)\mathbf{j} + F_3(x,y,z)\mathbf{k}$.

Suggestion: Use the Divergence Theorem or Stokes' Theorem to change the integration to a surface with $z = 0$. (Either one, properly used, will work.)

7. (12 points) Suppose $\mathbf{F} = \nabla\phi + \nabla \times \mathbf{G}$ in the domain $|\mathbf{R}| < 1$ and that $h(\mathbf{R})$ is a harmonic function for $|\mathbf{R}| < 1$. Let $\psi = \phi - h$. Derive a formula for $\mathbf{H}$ so that $\mathbf{F} = \nabla\psi + \nabla \times \mathbf{H}$ for $|\mathbf{R}| < 1$. The correct answer will include an integral over a single variable, but not an integral over a volume or a surface. You may leave that integral in your answer.

Suggestion: To start, equate the two expressions for $\mathbf{F}$ and rearrange.

Be sure to indicate where you need the fact that h is HARMONIC.

8. (12 points) In integral calculus, it was proved that, if S is an interval on the real axis that is symmetric about 0 and F is odd on S, then $\int_S F(x)\, dx = 0$. The goal of this exercise is to prove a similar theorem for line integrals.

A set S is symmetric about $\mathbf{0}$ if $\mathbf{R} \in S$ if and only if $-\mathbf{R} \in S$. A scalar or vector function F is $\left\{\begin{matrix}\text{even}\\ \text{odd}\end{matrix}\right\}$ on a set S if $F(-\mathbf{R}) = \left\{\begin{matrix}+F(\mathbf{R})\\ -F(\mathbf{R})\end{matrix}\right\}$ for all $\mathbf{R} \in S$. Recall that a regular curve does not cross itself.

$$\boxed{\text{Let } C \text{ be a regular curve symmetric about } \mathbf{0}.}$$

Fact (No need to prove it.): Such a curve C can be broken into two curves C_1 and C_2 such that $\mathbf{R} \in C_1$ if and only if $-\mathbf{R} \in C_2$. Furthermore, if $\mathbf{0} \notin C$, then the curve is closed.

Using this fact or otherwise, prove:

(a) If $\mathbf{0} \in C$ and $\mathbf{F}$ is <u>odd</u> on C, then $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{R} = 0$.

(b) If $\mathbf{0} \notin C$ and $\mathbf{F}$ is <u>even</u> on C, then $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{R} = 0$.

Remark: Drawing a picture for a simple case may help you see what is going on.
In (a), the picture could be a line segment like $\{(t,t) \mid -1 \le t \le 1\}$.
In (b), the picture could be a circle centered at the origin.

END OF EXAM