

1. There may be other answers than these
  - (a) gradient, scalar OR gradient, potential OR curl, solenoidal
  - (b) 0
  - (c) harmonic
  - (d) gradient, curl OR irrotational function, solenoidal function OR gradient of a scalar potential, curl of a vector potential

2. First solution: Use the divergence theorem and  $\nabla \cdot \nabla \times \mathbf{F} = 0$ .

Second solution: Use Stokes' Theorem. Since we are integrating over a closed surface, there is no boundary and so the integral is zero.

3. Since  $\mathbf{F}$  is defined everywhere, we can take the domain to be all of 3-space and  $\mathbf{R}_0 = \mathbf{0}$ .
  - (a) Since  $\mathbf{F}$  is homogenous of degree 2, the supplementary homework tells us that

$$\mathbf{G} = \frac{12}{2+2}(2xz\mathbf{i} - z^2\mathbf{k}) \times \mathbf{R} = 3(yz^2\mathbf{i} - 3xz^2\mathbf{j} + 2xyz\mathbf{k}).$$

Another way to solve it is to compute the integral in the text:

$$\begin{aligned} \mathbf{G} &= \int_0^1 t(24xzt^2\mathbf{i} - 12z^2t^2\mathbf{k}) \times \mathbf{R} dt = \int_0^1 t(12yz^2t^2\mathbf{i} - 36xz^2t^2\mathbf{j} + 24xyz t^2\mathbf{k}) dt \\ &= 3yz^2\mathbf{i} - 9xz^2\mathbf{j} + 6xyz\mathbf{k}. \end{aligned}$$

- (b) We can add the gradient of any function to  $\mathbf{G}$  without changing its curl. We need  $\phi$  such that  $(\mathbf{G} + \nabla\phi) \cdot \mathbf{k} = 0$ . The general solution to this is  $\phi = -3xyz^2 + f(x, y)$ . For simplicity, I took  $f = 0$ , but you could make another choice. This gave me  $\mathbf{G} + \nabla\phi = -12xz^2\mathbf{j}$ .

4. Looking at the given equations for  $D$ , we see that, in terms of  $u$  and  $v$ , it is the square

$$R = \{(u, v) \mid -1 \leq u \leq 1 \text{ and } -1 \leq v \leq 1\}.$$

We have

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2.$$

Since  $\frac{\partial(x, y)}{\partial(u, v)} \frac{\partial(x, y)}{\partial(u, v)} = 1$ , we have  $\partial(x, y)/\partial(u, v) = 1/-2 = -1/2$ . Alternatively, you could express  $x, y$  in terms of  $u, v$  and compute  $\partial(x, y)/\partial(u, v)$  directly. The answer can be written in various ways:

$$\iint_R v^2 e^{uv} \left| \frac{-1}{2} \right| dA = \frac{1}{2} \iint_R v^2 e^{uv} du dv = \frac{1}{2} \int_{-1}^1 \int_{-1}^1 v^2 e^{uv} du dv.$$

Aside: to evaluate the integral, use  $\int t e^{at} dt = \frac{1}{a} t e^{at} - \frac{1}{a^2} e^{at} + C$  and write

$$\begin{aligned} 2 \int_{-1}^1 \int_{-1}^1 v^2 e^{uv} du dv &= 2 \int_{-1}^1 v e^{uv} \Big|_{u=-1}^{u=1} dv = 2 \int_{-1}^1 (v e^v - v e^{-v}) dv \\ &= 2(v e^v - e^v + v e^{-v} + e^{-v}) \Big|_{v=-1}^{v=1} = 2(e^{-1} + e^{-1}) - 2(-e^{-1} - e^{-1}) = 8/e. \end{aligned}$$