1.
   - If $|x-2| > 1$, the series diverges since the terms do not go to zero.
   - If $|x-2| < 1$, the series converges absolutely by ratio test, or the root test or by comparison with the geometric series $\sum |x-2|^n$.
   - If $x-2 = +1$, the series is a divergent p-series.
   - If $x-2 = -1$, the series is alternating and converges since the terms decrease to zero.

   Putting all this together:

$$\begin{aligned} \text{conditional if and only if } & \ x = 1, \\ \text{absolute if and only if } & \ |x-2| < 1 \ \text{ that is, } \ 1 < x < 3. \end{aligned}$$

2. (a) The equation is linear: $y' + \dfrac{1}{x+1}y = \dfrac{2x}{x+1}$. You can use the formula for a linear equation or do it from "scratch." I'll do the latter. The integrating factor is $x+1$ and so $((x+1)y)' = 2x$. Thus $(x+1)y = x^2 + C$. Setting $x = 0$ and $y = 2$ gives $C = 2$. Thus $y = \dfrac{x^2+2}{x+1}$.
   Alternatively, the equation written as $(-2x+y)dx + (x+1)dy = 0$ is exact and one finds $-x^2 + xy + y = C$. The initial condition $(x, y) = (0, 2)$ gives $C = 2$.

   (b) This is an Euler equation. Even if you don't recognize it as such, you should recognize it as having a regular singular point and use the same approach. We try a solution of the form $y = x^r$. This leads to the indicial equation $r(r-1)+3r-3 = 0$ which has solutions $r = -3, 1$. The general solution is $y = c_1x^{-3} + c_2x$. From the initial conditions, $c_1 + c_2 = 4$ and $-3c_1 + c_2 = 0$. Solving: $c_1 = 1$ and $c_2 = 3$. Thus we have $y = x^{-3} + 3x$.

   (c) The homogeneous equation is $y''-4y = 0$. The characteristic equation is $r^2-4 = 0$ and so two independent solutions are $y_1 = e^{2t}$ and $y_2 = e^{-2t}$. Now we use variation of parameters. The Wronskian is $W(y_1, y_2) = -4$. By the formula for variation of parameters, a particular solution is

$$Y(t) = e^{2t} \int 4e^{-2t} \ln t \, dt - e^{-2t} \int 4e^{2t} \ln t \, dt.$$

   The general solution is $Y(t) + c_1e^{2t} + c_2e^{-2t}$.

3. The equilibrium points occur when $y = n\pi$ for some integer n. An equilibrium point of $y' = f(y)$ is stable when $f' < 0$ and unstable when $f' > 0$, so the unstable points are $y = 2n\pi$ and the stable ones are $y = (2n+1)\pi$. You were asked to find one of each.
4. Setting $y(x) = \sum a_nx^n$, we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} na_nx^n - \sum_{n=0}^{\infty} 2a_nx^n = 0,$$

which gives the recurrence relation $a_{n+2} = a_n/(n+1)$ for $n \geq 0$. From the initial conditions, $a_0 = 1$ and $a_1 = 0$. You can work out the required nonzero terms. I'll get the general term. Since $a_1 = 0$, the recurrence gives $a_n = 0$ for all odd n. For $n = 2k$,

$$a_{2k} = \frac{a_{2k-2}}{2k-1} = \frac{a_{2k-4}}{(2k-1)(2k-3)} = \cdots = \frac{1}{(2k-1)(2k-3)\cdots 3 \cdot 1}.$$

5. Let $y(t) = tv(t)$. Then $y' = tv' + v$ and $y'' = tv'' + 2v'$. Substitution and a bit of calculation gives

$$t^4 v'' + (2t^3 - t^2)v' = 0 \qquad \text{whence} \qquad \frac{dv'}{v'} = \frac{(-2t^3 + t^2)dt}{t^4}.$$

After integration, $\ln|v'| = -2\ln t - 1/t + C$. Exponentiating and choosing a convenient value for C:

$$v' = t^{-2}e^{-1/t} \qquad \text{whence} \qquad v = e^{-1/t},$$

by, for example, the substitution $-1/t = w$ to get $dw = dt/t^2$. Thus the general solution is $c_1 t + c_2 t e^{-1/t}$.

6. Since $x^2(1-x^2) = 0$ if and only if $x = 0,\ \pm 1$, these are the singular points. The point $x = 0$ is irregular because $xQ(x)/P(x)$ does not have a power series at $x = 0$. The other two are regular; e.g., for $x = 1$, both $(x-1)Q(x)/P(x)$ and $(x-1)^2 R(x)/P(x)$ have power series at $x = 1$.

7. Let $Y(s)$ be the Laplace transform of $y(t)$. From the table,

$$\mathcal{L}\{y'\} = sY - f(0) = sY - 1 \ \ \text{and} \mathcal{L}\{y''\} = s^2 Y - sf'(0) - sf(0) = s^2 Y - s.$$

What about $\mathcal{L}\{g(t)\}$? Many people forgot the definiton of the Laplace transform: $\mathcal{L}\{g(t)\} = \displaystyle\int_0^\infty e^{-st} g(t)\, dt$. Hence

$$\mathcal{L}\{g(t)\} = \int_0^2 e^{-st}\, dt + \int_2^\infty 0\, dt = -e^{-st}/s\Big|_{t=0}^{t=2} = \frac{1 - e^{-2s}}{s}.$$

Hence

$$s^2 Y - s - 2(sY - 1) + Y = \frac{1 - e^{-2s}}{s}.$$

Solving for Y, we get

$$Y = \frac{s - 2 + (1 - e^{-2s})/s}{s^2 - 2s + 1} = \frac{(s-1)^2 - e^{-2s}}{s(s-1)^2} = \frac{1}{s} - \frac{1}{s(s-1)^2 e^{2s}},$$

or any of a variety of equivalent forms.