1. (a) Separate variables: $\int e^y\,dy = \int -e^{-x}\,dx$ and so $e^y = e^{-x} + C$.

(b) Linear: $y' + y/t = e^t/t$ (provided $t \neq 0$). Integrating factor is $\exp\left(\int dt/t\right) = t$. Thus $(ty)' = e^t$ and so $ty = e^t + C$. Using the initial condition, $1 \times 1 = e^1 + C$, and so $C = 1 - e$. Finally

$$ty = e^t + 1 - e.$$

Alternate solution: $(y - e^t) + ty' = 0$ is exact since $M = y - e^t$ and $N = t$ give $M_y = N_x$.

(c) Homogeneous: Let $y = xv$ so $dy = x\,dv + v\,dx$. The given equation becomes

$$2x^2v(x\,dv + v\,dx) + (x^2 - x^2v^2)^2dx = 0.$$

After some algebra we have $2xv\,dv + (1+v^2)dx = 0$. Separate variables and integrate:

$$\int \frac{2v\,dv}{1+v^2} = \int \frac{-dx}{x}.$$

and so

$$\ln|1+v^2| = -\ln|x| + C; \quad \text{that is} \quad \ln(1 + y^2/x^2) + \ln|x| = C.$$

(Since $1 + v^2 > 0$, the first absolute value in not needed.) One can do some algebra and get the simpler form $x^2 + y^2 = Cx$. [If you think of x as a function of y, there's also the solution $x(y) = 0$, but you're not expected to find that.] Alternate solution: $(x^2 - y^2) + 2xyy' = 0$ has an integrating factor depending only on x since $M = x^2 - y^2$ and $N = 2xy$ give $(M_y - N_x)/N = -2/x$. (The integrating factor is $\mu = x^{-2}$.)

(d) The characteristic equation is $r^2 - 3r + 2 = 0$, which has roots 1 and 2. Hence the general solution to the homogeneous equation is $y = C_1e^t + C_2e^{2t}$. A particular solution to the nonhomogeneous equation can be found by undetermined coefficients. We try $y = a$. Substituting: $0 + 0 + 2a = 2$ and so $a = 1$. (Actually, the particular solution is so simple, you may have found it just by looking at the equation.) Thus the general solution is

$$y = C_1e^t + C_2e^{2t} + 1.$$

The initial conditions give $0 = y(0) = C_1 + C_2 + 1$ and $1 = y'(0) = C_1 + 2C_2$. Solving these equations gives $C_1 = -3$ and $C_2 = 2$. Hence we have

$$y = 1 - 3e^t + 2e^{2t}.$$

Alternate solution: Both this and (e) can be done by variation of parameters, but that would involve more work.

(e) This is the same equation as in (d). The initial conditions give $C_1 = C_2 = 0$ and so $y = 1$ is the solution.

2. (a) The general solution is

$$y = C_1 \cos(\omega t) + C_2 \sin(\omega t) \quad \text{or} \quad y = C_3 e^{i\omega t} + C_4 e^{-i\omega t}.$$

(b) It's easier to work with the trigonometric form. We have the two equations

$$0 = y(0) = C_1 \quad \text{and} \quad 0 = y(1) = C_1 \cos\omega + C_2 \sin\omega.$$

Thus $C_1 = 0$ and either $C_2 = 0$ or $\sin\omega = 0$. Since we want a solution different from zero, we cannot have $C_1 = C_2 = 0$ and so we must have $\sin\omega = 0$. In other words, ω must be a multiple of π.