1(a). $\sum \frac{(-1)^n}{n \ln n}$ converges: Alternating series whose terms decrease in magnitude.

1(b). $\sum \frac{1}{n \ln n}$ diverges: Integral test since $\int dx/(x \ln x) = \ln(\ln x) + C$.

1(c). $\sum \frac{n^9 + 100 \cos n}{\sqrt{n^3 + e^n}}$ converges: Easiest is ratio or root test since $a_{n+1}/a_n \to 1/e^{1/2}$.

1(d). $\sum \frac{\ln n}{n^2}$ converges: Easiest may be comparison test with a p-series where $1 < p < 2$; for example $p = 3/2$ works since $\ln n/n^2 < 1/n^{3/2}$ is equivalent to $\ln n < n^{1/2}$ which is true for large n since $\ln n$ grows slower than any positive power of n.

1(e). $\sum \cos n$ diverges since the terms do not go to zero as $n \to \infty$.

2. Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence of numbers and consider the series $S = \sum_{n=1}^{\infty} (a_{n+2} - a_n)$. Let S_N be the partial sum of the first N terms.
The general formula is $S_N = a_{N+2} + a_{N+1} - a_2 - a_1$. Hence S exists if and only if $\lim a_{N+2} + a_{N+1}$ exists. This will exist if $\lim_{n\to\infty} a_n = L$ exists and then
$S = 2L - a_2 - a_1$.
Remark: $\lim a_n$ need not exist for S to exist. For example, if $a_n = (-1)^n$, then $L = 0$ and $S = 0$.

3. $\sum \frac{n^2 - 2n}{3^n} (x - 5)^n$: Using the ratio test, $|a_{n+1}/a_n| \to |x - 5|/3$. Hence $R = 3$ and the ends of the interval are at $x - 5 = \pm 3$. At these values of x, the terms in the sum do not go to zero. Thus there is no convergence at the end points and so the interval is $(2, 8)$.

4. Since $f(x) = \cos x$, we have $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, $f^{(4)}(x) = \cos x$, and so on (the functions repeat after four derivatives). Since $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$, the Taylor series is $\sum_{n=0}^{\infty} \frac{\pm(x - \pi/4)^n}{\sqrt{2}\, n!}$ where the sign pattern is $+ - - +$ repeating. You can leave it at that or say the sign is plus when division of n by 4 has a remainder of 0 or 3 and minus otherwise. I would not expect anyone to do so, but you can actually get a formula for the sign: $(-1)^{n(n+1)/2}$.

5. Since $e^x = 1 + x + x^2/2 + x^3/6 + \cdots$ and $\tan^{-1} x = x - x^3/3 + \cdots$, we can take the product and throw away all terms higher that cubic:

$$\begin{aligned} e^x \tan^{-1} x &= (1 + x + x^2/2 + x^3/6)(x - x^3/6) + \cdots \\ &= (1 + x + x^2/2)x - (1)x^3/3 + \cdots = x + x^2 + x^3/6 + \cdots. \end{aligned}$$

Thus $c_0 = 0$, $c_1 = 1$, $c_2 = 1$, and $c_3 = 1/6$.