1. In the standard notation, $p(x) = 1/(1-x)(1+x)^2$ and $q(x) = 1/(1+x)(1-x)^2$.
   (a) Since $p(x)$ and $q(x)$ are okay except at $x = \pm 1$, these are the singular points. Regularity requires that $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ have power series at x_0. When $x_0 = 1$, they both have power series, so this point is regular. When $x_0 = -1$, $p(x)$ does not have a power series so this point is irregular (or you can say "not regular").
   (b) It converges: The power series for p and q about $x_0 = 0$ converge for $|x| < 1$. By a theorem in the book, so does the power series for $y(x)$.

2. We have $(s^2 Y(s) - s - 2) + (Y(s) - 1) = 1/s + 2/(s-1)$. Thus

$$Y(s) = \frac{1/s + 2/(s-1) + s + 3}{s^2+1} = \frac{s^3 + 2s^2 - 1}{s(s-1)(s^2+1)}.$$

3. Since the water level would rise 100 feet in 5 days, it is rising at the rate of 20 feet per day. This is the rate at which water is flowing in. Since it is flowing out at the rate of $5h^{1/2}$ feet per day, the differential equation is $h' = 20 - 5h^{1/2}$. The initial condition is $h(0) = 0$.

4. (a) The equation is linear: $xy' + 2y = 3x$. The integrating factor is x, so $(x^2 y)' = 3x^2$. Thus $x^2 y = x^3 + C$. Since $y(1) = 2$, $1^2 \times 2 = 1^3 + C$ and so $C = 1$. Hence $y = x + x^{-2}$.
   Alternatively, the equation is homogeneous
   (b) The equation is separable, so $\ln|y| = 3x - 2\ln|x| + C$. Since $y(1) = 2$, x y are positive and $\ln 2 = 3 + C$. Hence $C = \ln 2 - 3$. One can exponentiate to get a nicer form: $y = 2e^{3x-3}/x^2$.
   Alternatively, the equation is linear.
   (c) By undetermined coefficients, variation of parameters, or observation, $y = -t$ is a particular solution The homogeneous equation $y'' - y = 0$ has characteristic equation $r^2 - 1 = 0$ and so the general solution is $y = c_1 e^t + c_2 e^{-t} - t$. Using the initial conditions: $c_1 + c_2 = 0$ and $c_1 - c_2 - 1 = 0$. Thus $c_1 = 1/2$ and $c_2 = -1/2$. The equation can also be solved by Laplace transforms: $s^2 Y - Y = 1/s^2$. By algebra and partial fractions,

$$Y = \frac{1}{s^2(s^2-1)} = \frac{1/2}{s-1} - \frac{1/2}{s+1} - \frac{1}{s^2}.$$

   (d) The equation is homogeneous. Set $y = xv$ and $y' = xv' + v$ to obtain $xv' + v = 1 - v + v^2$. Thus $xv' = (1-v)^2$. Separate variables and integrate to get $(1-v)^{-1} = \ln x + C$. Thus $(1 - y/x)^{-1} = \ln x + C$. From the initial condition, $C = 1$. You can solve for y if you wish: $y = x - x/\ln(ex)$.

5. We set $y_2 = y_1 v = xv$. Since $y_2' = v + xv'$ and $y'' = 2v' + xv''$,

$$0 = x^3 y_2'' + x y_2' - y_2 = (2x^3 v' + x^4 v'') + (xv + x^2 v') - xv = x^4 v'' + (2x^3 + x^2)v'.$$

Separating variables:

$$\frac{dv'}{v'} = \frac{-(2x+1)dx}{x^2}.$$

Thus a particular solution is $\ln v' = -2\ln x + 1/x^2$. Exponentiating: $v' = x^{-2}e^{1/x}$. Integrating: $v = -e^{1/x}$. This gives $y_2 = -xe^{1/x}$ as an independent solution. (Since we can multiply by a constant, any solution of the form $y_2 = c_1 x e^{1/x} + c_2 x$ is acceptable if $c_1 \neq 0$.

6. We set $y = \sum a_n x^n$, differentiate it twice, substitute into the equation, and look at the coefficient of x^n to obtain the recursion

$$(n+2)(n+1)a_{n+2} - n(n-1)a_n + 4(n+1)a_{n+1} + 6a_n = 0.$$

Thus

$$a_0 = 1, \quad a_1 = -3, \quad \text{and} \quad a_{n+2} = \frac{(n^2 - n - 6)a_n - 4(n+1)a_{n+1}}{(n+2)(n+1)}.$$

With $n = 0$, $a_2 = 3$. With $n = 1$, $a_3 = -1$. With $n = 2$, $a_4 = 0$. With $n = 3$, $a_5 = 0$. From now on $a_{n+2} = 0$ since it depends only on the previous two values a_n and a_{n+1}.

7. (a) This is an Euler equation, so $y = x^r$ where $2r(r-1) + 3r - 1 = 0$. Thus $r = -1$ and $r = 1/2$ and so the general solution is $y = c_1/x + c_2 x^{1/2}$.

(b) Any method that produces a particular solution is acceptable. This includes using undetermined coefficients even though there is no reason that method should give a solution since undetermined coefficients is for *constant* coefficient linear equations.

The simplest approach is to note that, in solving Euler's equation, the left side is a function of r times x^r. Since we want to end up with x^2, we try $y = Cx^2$ for a particular solution. Then we get $4Cx^2 + 6Cx^2 - Cx^2 = 9x^2$. Hence $C = 1$ and so the general solution is $y = c_1/x + c_2 x^{1/2} + x^2$.

Alternatively, we can use the formula for variation of parameters (p. 176). Note that we must divide the given equation by $2x^2$ so that the coefficient of y'' is one. Thus $g(x) = 9/2$. After some calculations, $W(y_1, y_2) = 3/2(x^{3/2})$ and $Y = x^2$.

Alternatively, you can use the trick I mentioned for converting an Euler equation to a constant coefficient equation: Set $\ln x = t$. The given equation becomes $2d^2y/dt^2 + dy/dt - y = 9e^{2t}$. This can be solved in various ways. The general solution is $y = c_1 e^{-t} + c_2 e^{t/2} + e^{2t}$. Replace t with $\ln x$ to get the solution.