1. (a) An integrating factor is $e^{t/^2/2}$ and so $e^{t^2/2}y = \int t^3 e^{t^2/2}dt$. The integral can be done using integration by parts with $u = t$ or it can be done by substitution with $x = t^2/2$ and then integration by parts. The result is $(t^2 - 2)e^{t^2/2} + C$ and so $y = t^2 - 2 + Ce^{-t^2/2}$. Using the initial condition, $0 = -2 + C$ and so $C = 2$.

   (b) Separate variables: $e^{-y}dy = -t^{-2}dt$. Thus $-e^{-y} = t^{-1} + C$.

   (c) Set $y = e^{rt}$ to obtain $r^2 - 2r + 2 = 0$ and so $r = 1 \pm \sqrt{-1}$. Hence the general solution is $y = C_1 e^{(1+i)t} + C_2 e^{(1-i)t}$, which can be written $y = e^t(D_1 \sin t + D_2 \cos t)$.

   (d) The homogeneous equation $y'' - 4y' + y = 0$ gives $r^2 - 4r + 4 = 0$ and so $r = -2$ is a double root. Thus the general solution to the homogeneous equation is $y = (C_1 + C_2 t)e^{2t}$. One can use variation of parameters or undetermined coefficients to find a particular solution. Using the latter, we set $y = Ce^t$ and obtain $y'' - 4y' + 4y = Ce^t$ and so $C = 2$. Thus the general solution is $y = (C_1 + C_2 t)e^{2t} + 2e^t$.

   (e) In matrix notation, $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x}$. The determinant of $\begin{pmatrix} 2-r & -1 \\ 3 & -2-r \end{pmatrix}$ is $r^2 - 1$ and so $r = \pm 1$. Using $\mathbf{x} = e^t \mathbf{c}$ we obtain

$$e^t \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = e^t \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = e^t \begin{pmatrix} 2c_1 - c_2 \\ 3c_1 - 2c_2 \end{pmatrix},$$

   which has a solution $e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. With $\mathbf{x} = e^{-t}\mathbf{c}$ we obtain

$$-e^{-t} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = e^{-t} \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = e^{-t} \begin{pmatrix} 2c_1 - c_2 \\ 3c_1 - 2c_2 \end{pmatrix},$$

   which has a solution $e^{-t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Thus the general solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = d_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + d_2 e^{-t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

2. (a) Let $y = \sum a_n x^n$. Then

$$\begin{aligned} y'' - xy' - y &= \sum (n+1)(n+2)a_{n+2}x^n - \sum n a_n x^n - \sum a_n x^n \\ &= \sum (n+1)\big((n+2)a_{n+2} - a_n\big)x^n, \end{aligned}$$

   and so $a_{n+2} = \frac{a_n}{n+2}$ is the recursion.

   (b) We are given $a_0 = 1$ and $a_1 = 2$. It follows that

$$a_2 = \frac{a_0}{2} = \frac{1}{2}, \quad a_3 = \frac{a_1}{3} = \frac{2}{3}, \quad a_4 = \frac{a_2}{4} = \frac{1}{8}, \quad a_5 = \frac{a_3}{5} = \frac{2}{15}.$$

3. $\dfrac{d^2\theta}{dt^2} = \dfrac{d\omega}{dt} = \dfrac{d\omega}{d\theta}\dfrac{d\theta}{dt} = \dfrac{d\omega}{d\theta}\omega.$

(b) Separate variables and integrate to get $\omega^2/2 = K\cos\theta + C$.

4. $\mathcal{L}[y''(t)] = s^2Y(s) - s - 2, \quad \mathcal{L}[y(t)] = Y(s)$ and

$$\mathcal{L}[g(t)] = \int_0^\infty g(t)e^{-st}dt = \int_0^1 (1-t)e^{-st}dt$$
$$= \left.\frac{-(1-t)e^{-st}}{s}\right|_{t=0}^{1} - \frac{1}{s}\int_0^1 e^{-st}dt = \frac{1}{s} - \frac{1-e^{-s}}{s^2}.$$

Thus

$$Y(s) = \frac{\frac{1}{s} - \frac{1-e^{-s}}{s^2} + s + 2}{s^2 - 1} = \frac{s^3 + 2s^2 + s - 1 + e^{-s}}{s^2(s^2-1)}.$$