1. (a) Since $|\sin x| \le 1$, converges by comparison with $2\sum 1/n^2$ (the p test).
   (b) We have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{2|\sin n|}{n} &= \frac{\sin 1}{1}+\sum_{n=2}^{\infty}\frac{|\sin(n-1)|}{n-1}+\frac{|\sin n|}{n}\\ &\ge \sum_{n=2}^{\infty}\frac{|\sin(n-1)|+|\sin n|}{n}\\ &> \frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n}, \qquad \text{by } |\sin(x-1)|+|\sin x|>1/2.\end{aligned}$$

   Since $(1/2)\sum 1/n$ diverges to $+\infty$ and (b) is larger, it diverges.

2. The radius of convergence is 1. It converges conditionally for $x=2$ and absolutely for $0<x<2$.

3. (a) Separate variables: $e^{-x}dx = e^t dt$. Thus $-e^{-x} = e^t + C$ and we get $C=-2$.
   (b) The characteristic equation has roots $1\pm i$, so the general solution is

$$x = C_1e^t\cos t + C_2e^t\sin t.$$

   Since $x(0) = C_1$, we have $C_1 = 0$. Then $x' = C_2e^t(\sin t+\cos t)$ and, since $x'(0)=2$, $C_2=2$. In summary, $x = 2e^t\sin t$.
   (c) The equation is exact: $\partial(2x+y)/\partial y = \partial(3y^2+x)/\partial x$. Since

$$\int(2x+y)dx = x^2+xy+f(y) \quad\text{and}\quad \int(3y^2+x)dy = y^3+xy+g(x),$$

   the general solution is $x^2+xy+y^3=C$.
   (d) If we think of z as a function of w, the equation is linear: $z+w = w\,dz/dw$. Writing in the standard way: $z'(w)-(1/w)z = 1$. Since $\exp\left(\int -(1/w)dw\right) = 1/w$, $(z/w)' = 1/w$ and so $z = w\int(1/w)dw = w(\ln w + C)$.
   (e) The indicial equation for $x^2y''-6y=0$ is $r(r-1)-6=0$ and so $r=3,-2$. Thus the general solution to the homogeneous equation is $y = C_1x^{-2}+C_2x^3$. The Wronskian is $\begin{vmatrix} x^{-2} & x^3 \\ -2x^{-3} & 3x^2\end{vmatrix} = 5$. By variation of parameters for the equation $y''-(6/x^2)y=5$, a particular solution is

$$\begin{aligned}-x^{-2}\int\frac{x^3 5}{5}dx + x^3\int\frac{x^{-2}5}{5}dx &= -x^{-2}\int x^3dx + x^3\int x^{-2}dx\\ &= -x^{-2}\frac{x^4}{4}+x^3(-x^{-1}) = -5x^2/4.\end{aligned}$$

   Thus the general solution is $y = C_1x^{-2}+C_2x^3-5x^2/4$.

4. The Wronskian of two independent solutions t and e^t must be nonzero for all t inside the interval where the solutions exist; in this case for all t. The Wronskian of these two functions is $(t-1)e^t$, which vanishes at $t = 1$.

5. Separating variables: $h^{-1/2}dh = -2dt$ and so $2h^{1/2} = -2t + C$. Since $h(0) = 16$, $C = 2\sqrt{16} = 8$. Thus $h^{1/2} = 4 - t$.
   (a) After 1 hour, $h^{1/2} = 3$ and so $h = 9$ — the water is nine feet deep.
   (b) After 6 hours, one might (incorrectly) write $h^{1/2} = 4-6 = -2$ and conclude that the water is four feet deep. In fact, at $t = 4$ we had division by zero in separation of variables. Thus our solution is only valid for $0 \le t \le 4$. After four hours the tank is empty and remains so.

6. We have $\sum(n+1)(n+2)a_{n+2}x^n - \sum na_nx^n - \sum a_nx^n = 0$. Thus the recursive relation is
$$(n+1)(n+2)a_{n+2} - (n+1)a_n \;=\; 0 \quad \text{and so} \quad a_{n+2} \;=\; \frac{a_n}{n+2}.$$
The initial conditions give us $a_0 = y(0) = 1$ and $a_1 = y'(0) = 1$. Thus
$$a_2 = \frac{a_0}{2} = \frac{1}{2}, \quad a_3 = \frac{a_1}{3} = \frac{1}{3}, \quad a_4 = \frac{a_2}{4} = \frac{1}{8}.$$

7. Since $s^2Y(s) - sy(0) - y'(0) + 2Y(s) \;=\; G(s)$, $Y(s) \;=\; \frac{1+G(s)}{2+s^2}$. We can compute $G(s)$ in one of two ways:
   (a) By integration:
$$G(s) \;=\; \int_0^1 e^{-st}(+1)\,dt + \int_1^2 e^{-st}(-1)\,dt \;=\; \frac{1-2e^{-s}+e^{-2s}}{s}.$$
   (b) By using $u_c(t)$ and the table of Laplace transforms:
$$g(t) \;=\; 1 - 2u_1(t) + u_2(t) \quad \text{and so} \quad G(s) \;=\; \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s}.$$
   Thus we have
$$Y(s) \;=\; \frac{s+1-2e^{-s}+e^{-2s}}{s(2+s^2)}.$$