1. While one could differentiate five times, it is easier to multiply series. In the following, $\cdots$ stands for terms of degree higher than five since we only need to keep terms through the fifth degree.

$$\begin{aligned}\cos x &= 1 - x^2/2 + x^4/24 + \cdots \\ \sin(x^2) &= x^2 - \cdots \\ f(x) &= (1 - x^2/2 + x^4/24 - \cdots)(x^2 - \cdots) \;=\; x^2 - x^4/2 + \cdots\end{aligned}$$

   Thus $a_1 = a_3 = a_5 = 0$, $a_2 = 1$ and $a_4 = -1/2$.

2. (a) The ratio test shows that the series converges for all x. ($R = \infty$)
   (b) The root test gives $|2x - 3| < 1$, which is equivalent to $|x - 3| < 1/2$, so the radius of convergence is $1/2$.
   (c) The ratio or root test has a limit of $|x|$ and so $R = 1$.

3. Often more than one test can be used.
   (a) The series diverges because the terms do not go to zero:
   $|a_n| = (9/8)^n/108n^2 \to \infty$.
   You could also use the ratio or the root test.
   (b) It converges. Perhaps the easiest test to use in the integral test with
   $f(x) = x^{-1/2}e^{-x^{1/2}}$ since $\int f(x)\,dx = 2e^{-x^{1/2}} + C$.
   (c) Since $\sin x$ is an increasing function when x is small, $\sin(1/n)$ is a decreasing function. By the alternating series test, we have convergence. Comparison with $\sum 1/n$ shows that we do not have absolute convergence.
   (d) The ratio test has a limit of $1/4$ and so the series converges.

4. (a) Using the series for $f(x)$: $f(-x) = \sum a_n(-x)^n = \sum(-1)^n a_n x^n$.
   Thus $b_n = (-1)^n a_n$
   (b) Since power series are unique and $f(-x) = f(x)$, the two series have the same coefficients; that is, $b_n = a_n$. From (a), $a_n = (-1)^n a_n$, which gives us $a_n = -a_n$ for odd n. Thus $a_n = 0$ when n is odd.