1. Let $f(x,y) = 4 - x^2 - y^2$. The xy-plane is given by $z = 0$. Thus the intersection with $z = 4 - x^2 - y^2$ is given by $x^2 + y^2 = 4$. Since $f_x = -2x$ and $f_y = -2y$, the area is

$$\iint\limits_D \sqrt{1 + 4x^2 + 4y^2}\, dA \quad \text{where} \quad D = \{(x,y) \mid x^2 + y^2 \le 4\}.$$

We must convert this to an iterated integral. This can be done in Cartesian coordinates in two ways:

$$\int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \sqrt{1 + 4x^2 + 4y^2}\, dx\, dy = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{1 + 4x^2 + 4y^2}\, dy\, dx$$

and in polar coordinates in two ways:

$$\int_{\alpha}^{\alpha+2\pi} \int_{0}^{2} \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \int_{0}^{2} \int_{\alpha}^{\alpha+2\pi} \sqrt{1 + 4r^2}\, r\, d\theta\, dr,$$

where your answer can have any value for α; e.g., 0 or $-\pi$.

2. (a) $r = \sqrt{1^2 + 3} = 2$, $\theta = \pi/3$ and $z = 2$.
   (b) $\rho = \sqrt{1^2 + 3 + 2^2} = \sqrt{8}$, $\theta = \pi/3$ and $\phi = \pi/4$.

3. (a) Any vector $c\langle 1,1,0\rangle \times \langle 0,1,2\rangle = c\langle 2,-2,1\rangle$ with $c \ne 0$.
   (b) Since $\langle 0,0,0\rangle$ is on the first line and $\langle 1,1,1\rangle$ is on the second, the closest distance is given by the length of the projection of $\mathbf{v} = \langle 1,1,1\rangle$ onto $\mathbf{w} = \langle 2,-2,1\rangle$. This equals
   $$\frac{|\mathbf{u} \cdot \mathbf{w}|}{|\mathbf{w}|} = \frac{|2 - 2 + 1|}{\sqrt{4 + 4 + 1}} = 1/3.$$

4. This is an example in the text. The answer is $f(0, \pm 1) = 2$ (maxima) and $f(\pm 1, 0) = 1$ (minima).

5. (a) $(\mathbf{f}(t) \cdot \mathbf{g}(t))' = \mathbf{f}'(t) \cdot \mathbf{g}(t) + \mathbf{f}(t) \cdot \mathbf{g}'(t)$ At $t = 2$ this is 1.
   (b) $|\mathbf{f}(t)|' = \left(\sqrt{\mathbf{f}(t) \cdot \mathbf{f}(t)}\right)' = \dfrac{\mathbf{f}'(t) \cdot \mathbf{f}(t) + \mathbf{f}(t) \cdot \mathbf{f}'(t)}{2\sqrt{\mathbf{f}(t) \cdot \mathbf{f}(t)}}$. At $t = 2$ this equals $-1/\sqrt{5}$.
   (c) Since $\mathbf{v} \times \mathbf{v} = 0$ for any vector $\mathbf{v}$, $(\mathbf{f}(t) \times \mathbf{f}(t))$ is constant—the zero vector. Thus its derivative is the zero vector $\langle 0,0,0\rangle$. (Since it is *not* the scalar 0, you will lose some points if you write the scalar 0 instead of the vector $\vec{0}$.)

6. You'll have to imagine the sketch using the following description. The region lies in the first quadrant, is bounded below by the parabola $y = x^2$ and above by the line $y = 2x$. These curves intersect at $(0,0)$ and $(2,4)$. When integrating in the other order, x goes from $y = 2x$ to $y = x^2$ and then y goes from 0 to 4. Thus the answer is

$$\int_{0}^{4} \int_{y/2}^{\sqrt{y}} f(x,y)\, dx\, dy.$$

7. Since $h_x = f$ and $h_y = g$, it follows that $h_{xy} = f_y$ and $h_{yx} = g_x$. Since $h_{xy} = h_{yx}$ (Clairaut's Theorem), we have $f_y = g_x$.

8. The intersection with the xy-plane is the circle $x^2 + y^2 = 4$. Thus the answer is

$$\int_0^{2\pi} \int_0^2 (4 - r^2) r \, dr \, d\theta = \int_0^{2\pi} \Big(2r^2 - r^4/4\Big]_{r=0}^{r=2} d\theta = \int_0^{2\pi} 4 \, d\theta = 8\pi.$$

9. This is most easily done by writing it as a sum of two iterated integrals:

$$\int_0^1 \int_0^2 xe^{xy} \, dy \, dx + \int_0^2 \int_0^1 ye^{xy} \, dx \, dy.$$

The first integral is

$$\int_0^1 \int_0^2 xe^{xy} \, dy \, dx = \int_0^1 \Big(e^{xy}\Big]_{y=0}^{y=2} dy \, dx = \int_0^1 (e^{2x} - 1) \, dx = \Big(\tfrac{1}{2}e^{2x} - x\Big]_0^1 = e^2/2 - 3/2.$$

Similarly,

$$\int_0^2 \int_0^1 ye^{xy} \, dx \, dy = \int_0^2 (e^y - 1) \, dy = e^2 - 3.$$

Combining these we have the answer: $3e^2/2 - 9/2$.

If you do not split the integral into two, it is still possible to do it, but it is quite a bit more work. Suppose we integrate over x and then y. Using integration by parts

$$\begin{aligned}\int (x+y)e^{xy} \, dx &= \int xe^{xy} \, dx + \int ye^{xy} \, dx \\ &= x(1/y)e^{xy} - \int (1/y)e^{xy} \, dx + e^{xy} = (x/y - 1/y^2 + 1)e^{xy}.\end{aligned}$$

Thus $\int_0^1 (x+y)e^{xy} \, dx = (1/y - 1/y^2 + 1)e^y + 1/y^2 - 1$. Using integration by parts with $u = 1/y$ and $dv = e^y \, dy$, we have

$$\int (1/y)e^y \, dy = (1/y)e^y + \int (1/y^2)e^y \, dy.$$

Thus

$$\begin{aligned}\int \Big((1/y - 1/y^2 + 1)e^y + 1/y^2 - 1\Big) \, dy &= (1/y)e^y + \int (e^y + 1/y^2 - 1) \, dy \\ &= (1/y)e^y + e^y - 1/y - 1.\end{aligned}$$

The integral from $y = 0$ to $y = 2$ is improper and can be evaluated because $\lim_{y\to 0}(e^y - 1)/y = 1$ since it is the definition of the derivative of e^y at $y = 0$. The rest is straightforward and we get the same answer as before.