Each quiz is worth 12 points.

Q1. Each part is worth 6 points.

(a) The first is an application of the Fundamental Theorem of Calculus and the rule for differentiating composition of functions. The answer is

$$\sqrt{(x^2)^3+1}\,\frac{dx^2}{dx} \;=\; 2x\sqrt{x^6+1}.$$

There are 3 points for each factor; that is, 3 points for $2x$ and three points for $\sqrt{(x^2)^3+1}$ (which can be written $\sqrt{x^6+1}$). If you differentiate x^2 incorrectly, there are no points for that factor. If you get $\sqrt{x^3+1}$ or some other incorrect factor, there are no points for that factor.

(b) This is almost identical to a homework problem (5.5 #63). Use the substitution $u = t-1$ to obtain (remembering that $du = dt$)

$$\begin{aligned}\int t\sqrt{t-1}\,dt \;&=\; \int (u+1)\sqrt{u}\,du \;=\; \int (u^{3/2}+u^{1/2})\,du \\ &=\; \frac{2u^{5/2}}{5}+\frac{2u^{3/2}}{3}+C \;=\; \frac{2(t-1)^{5/2}}{5}+\frac{2(t-1)^{3/2}}{3}+C.\end{aligned}$$

Of course, you might not choose to call the new variable u.

1 point: giving correct substitution;
1 point: getting $du = dt$;
1 point: doing the substitution correctly;
1 point: evaluating the resulting integral;
1 point: getting the final result in terms of t;
1 point: having $+C$.

If you make an error somewhere along the line, you will lose the one point for that step. If the remainder of the work based on that error is done correctly, you will receive credit for those steps.
Exception: If you write $\sqrt{t-1} = t-1$, or $\sqrt{t}-1$, or some similar misuse of square root, you will lose *all* subsequent points except possibly $+C$.

Subsequent quizzes may not give such a detailed point break-down. Also, as we get more material, subsequent quizzes are likely to be more difficult.

Q2. The region is bounded above by the parabola and below by the line. The curves intersect at (0,0) and (3,3).

(a) $A \;=\; \displaystyle\int_0^3 (x(4-x)) - x)\,dx \;=\; \int_0^3 (3x-x^2)\,dx \;=\; \left(\frac{3x^2}{2}-\frac{x^3}{3}\right]_0^3 \;=\; 3^3/2-3^2.$

You need not carry out the arithmetic to get 9/2.

4 points for setting up the integral correctly;
2 points for the integration.

(b) We need to get x as a function of y, so we have $x = y$ and $x = 2 - \sqrt{4-y}$. The latter comes from $y = 4x - x^2 = 4 - (x-2)^2$ and the correct choice of the sign on the square root. Thus the answer is

$$\pi \int_0^3 \left(y^2 - (2 - \sqrt{4-y})^2\right) dy \;=\; \pi \int_0^3 y^2\, dy - \pi \int_0^3 (2 - \sqrt{4-y})^2\, dy.$$

2 points to realize you must express x as a function of y;
2 points for solving correctly for x for the parabola;
2 points for writing down the integral correctly, based on whatever you found when solving the parabola for x.

Some people may have used the shell method for computing the volume. In that case, you would have two integrals depending on whether the top boundary was the parabola or $y = 3$. To get the intersection: $3 = 4 - (x-2)^2$ and use the quadratic formula. Thus $x = 1$. The answer is

$$2\pi \int_0^1 x((4x - x^2) - x)\, dx + 2\pi \int_1^3 x(3 - x)\, dx.$$

2 points for realizing there are two integrals and finding $x = 1$;
2 points for writing down each integral correctly, based on whatever limits you found if you did not get $x = 1$.

Q3. (a) Since $x^2 - 1 = (x-1)(x+1)$ and $x^2 + 1$ does not factor and the degree of the numerator is less than the degree of the denominator, the answer is

$$\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{Ex+F}{x^2+1}.$$

Of course, the order of the terms and the names of the constants A–F are irrelevant.
1 point for the A and C terms.
1 point for the B and D terms.
2 points for the $Ex + F$ term. (1 each for Ex and F).

There is an alternate form which is acceptable:

$$\frac{Ax+B}{(x-1)^2} + \frac{Cx+D}{(x+1)^2} + \frac{Ex+F}{x^2+1}.$$

In that case there are 2 points for recognizing that all of A–D must be present. There is no partial credit for these two points.

(b) Let $x = \sec t$ (and so $dx = \sec t\, \tan t\, dt$) to get

$$\int \frac{dx}{x^2\sqrt{x^2-1}} \;=\; \int \frac{\sec t\, \tan t\, dt}{\sec^2 t\, \tan t} \;=\; \int \cos t\, dt \;=\; \sin t + C \;=\; \frac{\sqrt{x^2-1}}{x} + C.$$

1 point for the correct substitution.

1 point for getting to $\int \cos t\, dt$.

1 point $+C$.

1 point for correctly substituting back into whatever answer you got to get an expression in terms of x.

(c) Let $t = e^x$ (and so $x = \ln t$ and $dx = dt/t$) to get

$$\begin{aligned}\int \frac{dx}{e^x+1} &= \int \frac{dt}{t(t+1)} = \int \left(\frac{1}{t} - \frac{1}{t+1}\right) dt \\ &= \ln|t| - \ln|t+1| + C = x - \ln(e^x+1) + C.\end{aligned}$$

You may rewrite this as $-\ln(1+e^{-x}) + C$, $-\ln|1+e^{-x}| + C$, or something else that is equivalent — but there is no need to do so.

1 point for the substitution $t = e^x$, or some other substitution (such as $t = e^{-x}$) that gives a rational integrand.

1 point for correct partial fraction decomposition of your integrand.

1 point for correctly integrating your partial fraction decomposition.

1 point for getting the final answer. This point is all or nothing, except you get the point if $+C$ is missing. No point if you have $|x|$ instead of x. However, it is okay to have $\ln|e^x+1|$ since $e^x + 1 > 0$ for all x.

Q4 (a) Since $h = (b-a)/n = (1-0)/3 = 1/3$, we have

$$\tfrac{1}{6}\left(1+2\sqrt{1-\left(\tfrac{1}{3}\right)^3}+2\sqrt{1-\left(\tfrac{2}{3}\right)^3}+0\right) \quad \text{or} \quad \tfrac{1}{3}\left(\tfrac{1}{2}+\sqrt{1-\left(\tfrac{1}{3}\right)^3}+\sqrt{1-\left(\tfrac{2}{3}\right)^3}+0\right)$$

where, of course, you can leave off the "+0".

1 point for getting the interval (1/3) correct.

1 point for knowing that the integrand must be evaluated at 0, 1/3, 2/3 and 1.

1 point for knowing that the end values are multiplied by $h/2$ rather than h.

Since the problem was vague about the form of the solution, you could also write the answer as

$$\frac{1}{t}\Big(f(0) + 2f(1/3) + 2f(2/3) + f(1)\Big) \quad \text{where} \quad f(x) = \sqrt{1-x^3}.$$

If you do not say what $f(x)$ is, you **lose 1 point** because there no value for $f(x)$ was given in the problem. Otherwise, points are calculated as indicated above.

(b) (3 points) It is convergent. You can do this by evaluating the integral:

$$\int_4^\infty \frac{dx}{(x-1)^2} = \lim_{a\to\infty}\left[\frac{-1}{x-1}\right]_4^a = \lim_{a\to\infty}\left(\frac{1}{3} - \frac{1}{a-1}\right) = \frac{1}{3}.$$

You lose 2 points if you do not rewrite the improper integral as a limit.

You can do this by a change of variable: $\int_4^\infty \frac{dx}{x-1)^2} = \int_3^\infty \frac{dt}{t^2}$, which we know converges by the test in the text since $2 > 1$.
You can do this by the comparison test. That is a bit tricky: in $\frac{1}{(x-1)^2} > \frac{1}{x^2}$, the inequality is in the wrong direction. It can be shown that $\frac{1}{(x-1)^2} \le \frac{16}{9x^2}$, but I'll omit the details since I doubt that anyone has done this correctly.

(c) (3 points) It diverges since $\frac{1+e^{-x}}{x} > \frac{1}{x}$ and $\int_1^\infty \frac{dx}{x}$ diverges.

(d) One point for each correct identification. $I = 5.05$, $R_{20} = 5.06$, $T_{20} = 5.02$.

If you're interested, here's one way to reach this conclusion. Since the function is increasing ($f' > 0$), R_{20} is greater than all the other estimates we studied. Since the function is convex ($f'' > 0$), the Midpoint Rule underestimates the integral. (You can also see how R and T behave by drawing a picture.)

Q5. (a) We have $y' = b^2$ and so $(xy' - y)^2 = (2b)^2 = 4y'$. Thus the answer is all b.
4 points for getting all b; otherwise no points.

(b) We have $y' = -c/x^2$ and so $(xy' - y)^2 = (-c/x - c/x)^2 = 4c^2/x^2$. For this to equal $4y'$ we must have $4c^2 = -4c$. Thus the answer is $c = 0$ and $c = -1$.
If you say all c, no points.
2 points each for $c = 0$ and $c = -1$, but if you have any answers other than these, you lose 2 points, but don't go negative. For example, the answer $c = 0$, $c = 1$ and $c = -1$ is worth $4 - 2 = 2$ points; the answer $c = \pm 1$ is worth $2 - 2 = 0$ points; the answer $c = 5$ is worth no points.

(c) Since we are rotating about the y-axis, the area is $\int 2\pi x \sqrt{(dx)^2 + (dy)^2}$. At this point we can proceed in either of two directions:

• If we integrate over y, then $x(y) = (64/y^3)^{1/2} = 8y^{-3/2}$ and$x'(y) = -12y^{-5/2}$. Thus the answer is

$$2\pi \int_1^4 8y^{-3/2}\sqrt{1 + 12^2/y^5}\, dy \;=\; 16\pi \int_1^4 y^{-3/2}\sqrt{1 + 144/y^5}\, dy.$$

• If we integrate over x, then $y(x) = (64/x^2)^{1/3} = 4x^{-2/3}$ and
$y'(x) = -(8/3)x^{-5/3}$. Thus the answer is

$$2\pi \int_1^8 x\sqrt{1 + (8/3)^2/x^{10/3}}\, dx \;=\; \frac{16\pi}{3} \int_1^8 x\sqrt{1 + 64/9x^{10/3}}\, dx.$$

1 point for knowing the correct form of the integral, either
$2\pi \int x\sqrt{(dx)^2 + (dy)^2}$, $2\pi \int x\sqrt{1 + (y'(x))^2}\, dx$ or $2\pi \int x(y)\sqrt{1 + (x'(y))^2}\, dy$.
1 point for solving for $x(y)$ or $y(x)$ correctly, with or without doing the arithmetic. For example, $y = (64/x^2)^{1/3}$ is fine.

1 point for having the final integral written down correctly, doing the arithmetic.

Q6. (a) Since $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$ and $\cos(2x) = \frac{e^{2ix}+e^{-2ix}}{2}$, The answer is

$$\frac{e^{ix}-e^{-ix}}{2i}\frac{e^{2ix}+e^{-2ix}}{2} = \frac{e^{3ix}-e^{ix}+e^{-ix}-e^{-3ix}}{4i},$$

which could be rearranged in various ways.
Alternatively, one could use the identity for a product of sine and cosine to write

$$\sin x\,\cos(2x) = \frac{\sin(3x)-\sin x}{2} \qquad \text{or, equivalently,} \qquad \frac{\sin(3x)+\sin(-x)}{2}$$

and then convert this to exponentials.

1 point each for knowing how to write $\sin x$ and $\cos(2x)$ in terms of exponentials or, in the second method, $\sin(3x)$ and either $\sin(-x)$ or $\sin x$.

2 points for getting the final answer correct.

(b) Since $x^2+4 = (x+2i)(x-2i)$, the form is $\frac{A}{x+2i}+\frac{B}{x-2i}$. By whatever method you choose, $A = B = 1$.

2 points for knowing the form.

1 point each for A and B.

(c) Since $-1 = e^{x+iy} = e^x\cos y + (e^x \sin y)i$, we see that $\sin y = 0$ and so $\cos y = \pm 1$. To get $e^x \cos y = -1$, we must have $x = 0$ and $\cos y = -1$. Thus $y = \pi$ plus any multiple of 2π. In other words, any *odd* multiple of π is an acceptable answer.

2 points for $x = 0$

2 points for $y = \pi$ or $y = -\pi$ or $y = 3\pi$, etc. (1 point if you realize that $\sin y = 0$ but do not get y.)