1. (a) By parts:

$$\int_1^e x^3 \ln x\, dx \;=\; (x^4/4)\ln x\Big|_1^e - \int_1^e \frac{x^3}{4}\,dx \;=\; \Big((x^4/4)\ln x - x^4/16\Big|_1^e \;=\; 3e^4/16 + 1/16.$$

   (b) Since $(2+\cos x)/x \ge 1/x$ and $\int_0^{\pi/2}(1/x)\,dx$ diverges, so does the given integral.

   (c) Let $1+2u = t$ to obtain $\int t^10\,(dt/2) \;=\; t^11/22 + C \;=\; (1+2u)^11/22 + C$.

   (d) With $x = \sin t$ and $dx = \cos t\, dt$,

$$\int_0^{1/\sqrt{2}} \frac{x^2}{\sqrt{1-x^2}}\,dx \;=\; \int_0^{\pi/4} \sin^2 t\, dt \;=\; \int_0^{\pi/4} \frac{1-\cos 2t}{2}\,dt$$
$$=\; \frac{t-(\sin 2t)/2}{2}\Big|_0^{\pi/4} \;=\; \pi/8 - 1/4.$$

   (e) By partial fractions,

$$\int_1^2 \frac{1}{u+u^2}\,du \;=\; \int_1^2 \left(\frac{1}{u} - \frac{1}{1+u}\right)du$$
$$=\; \Big(\ln|u| - \ln|1+u|\Big|_1^2 \;=\; (\ln 2 - \ln 3) - (-\ln 2) \;=\; \ln(4/3).$$

   (f) With $e^t = u$, $\int e^{t+e^t}\,dt \;=\; \int e^u\,du \;=\; e^u + C \;=\; e^{e^t} + C$.

2. (a) $w \;=\; \dfrac{1+i}{1-3i}\,\dfrac{1+3i}{1+3i} \;=\; \dfrac{-2+4i}{10} \;=\; \dfrac{-1}{5} + \dfrac{2i}{5}$. and $\overline{w} \;=\; \dfrac{-1}{5} - \dfrac{2i}{5}$.

   (b) In polar form, $1-i$ is $r = \sqrt{2}$, $\theta = -\pi/4$. Thus the three answers have $r = 2^{1/6}$ and $\theta \;=\; -\pi/12,\ 7\pi/12,\ 5\pi/4$.

   (c) $e^2/\sqrt{2} + e^2\sqrt{2}\,i$.

3. Notice that the values of x range from -1 to $+1$. As remarked on the exam, $y \;=\; \pm(1-x^4)^{1/4}$ and so $y' \;=\; \pm(-4x^3)(1-x^4)^{-3/4}/4 \;=\; \mp x^3(1-x^4)^{-3/4}$. The region is symmetric in all four quadrants, so one can integrate over one quadrant and multiply by 4 for (a) and (b). One could also integrate over the upper or right half-plane and double or over the whole plane, so there are various forms for the answers.

   (a) (whole plane) $\displaystyle\int_{-1}^{1}\Big((1-x^4)^{1/4} - (-(1-x^4)^{1/4})\Big)\,dx$

   (b) (twice upper half-plane) $\displaystyle 2\int_{-1}^{1}\sqrt{1+(y')^2}\,dx \;=\; 2\int_{-1}^{1}\sqrt{1+x^6\sqrt{1-x^4}}\,dx$

(c) (twice right half-plane) $2\int_0^1 \left(\pi(y_{\text{up}}+2)^2 - \pi(y_{\text{down}}+2)^2\right) dx$

$= 2\pi \int_0^1 \left(((1-x^4)^{1/4}+2)^2 - (-(1-x^4)^{1/4}+2)^2\right) dx \; = \; 16\pi \int_0^1 (1-x^4)^{1/4}\, dx$

(You need not simplify as was done here.)

4. Rewriting in a more standard form and identifying:

(a) $-2x^2 + y^2 \; = \; 3$, a hyperbola;

(b) $\dfrac{1}{1+(1/2)\cos\theta}$, an ellipse;

(c) $2x^2 + y^2 \; = \; 3$, an ellipse.

5. Division by zero can be a problem.

(a) Separating variables would give the general solution of $-1/x \; = \; t + C$ and so $x \; = \; \frac{-1}{t+C}$; however, this involves division by zero when $x = 0$ and does not include the particular solution to this problem, namely $x(t) = 0$ for all t.

(b) Rearranging: $y' \; = \; (1-y)/x$ and so, after separating variables, $-\ln(1-y) \; = \; \ln x + C$. You can leave the solution this way or you can solve for y: $y \; = \; 1 + A/x$, where we have replaced the constant e^{-C} with the constant A. There is a problem with division by zero when separating variables, but you will not lose any points if you ignored it. The division by zero solution is given by $y = 1$, which is not included in $-\ln(1-y) \; = \; x + C$, but is included in $y \; = \; 1 + A/x$. It is not included in the first form because that would require $C = \infty$, but it is included in the second form with $A = 0$. How did this happen? We have $A = e^{-C}$, which would not let $A = 0$ unless $C = \infty$.

6. The curves intersect at $\cos\theta = 0$; that is $\theta = \pm\pi/2$. Since $1 + \cos\theta > 1$ for $-\pi/2 < \theta < \pi/2$, we obtain

$$\int_{-\pi/2}^{\pi/2} \frac{(1+\cos\theta)^2 - 1^2}{2}\, d\theta.$$

Had we noticed the symmetry of the region, we could have integrated from 0 to $\pi/2$ and doubled the result:

$$\int_0^{\pi/2} \left((1+\cos\theta)^2 - 1^2\right) d\theta.$$

THERE ARE MORE PROBLEMS

7. (a) Since $t/2 = \arctan x$, $\dfrac{dt}{dx} = \dfrac{2}{1+x^2}$.

   (b) $\sin x = \dfrac{\frac{1+ix}{1-ix} - \frac{1-ix}{1+ix}}{2i} = \dfrac{(1+ix)^2 - (1-ix)^2}{2i(1-ix)(1+ix)} = \dfrac{2x}{1+x^2}$ and similarly

   $\cos x = \dfrac{1-x^2}{1+x^2}$.

8. The ordering is $R_{100} < T_{100} < I < M_{100} < L_{100}$. To see this, let $f(x) = e^{-x^2/2}$. We have $f'(x) = -xe^{-x^2/2}$, which is negative for $0 < x < 1$ and $f''(x) = (x^2-1)e^{-x^2/2}$, which is also negative for $0 < x < 1$. At this point one can draw a picture of $f(x)$, sketch one interval, and see that the order is correct for that interval. Alternatively, one could state: From the f' result, R_{100} is smaller than all other values and L_{100} is larger than all other values. From the f'' result, $T_{100} < I < M_{100}$.

END OF EXAM