Math 20B Second Midterm Solutions November 20, 2003

1. Since (a) is not an improper integral, it cannot diverge. We can integrate (b):

$$\lim_{a\to 0+}\int_a^1 x^{-1/2}\,dx \;=\; \lim_{a\to 0}(1-a^{1/}) \;=\; 1.$$

Thus it converges.

2. (a) Since $y' = 1/x$, we have $\int_1^e \sqrt{1+x^{-2}}\,dx$.

(b) The integral is $\int_{x=1}^{x=e} 2\pi x\sqrt{(dx)^2+(dy)^2}$. You can write this as

$$\int_1^e 2\pi x\sqrt{1+(dy/dx)^2}\,dx \;=\; \int_1^e 2\pi x\sqrt{1+x^{-2}}\,dx,$$

or you can notice that $x = e^y$ and $dx/dy = e^y$ and thus write it as

$$\int_0^1 2\pi e^y\sqrt{1+e^{2y}}\,dy.$$

3. (a) Since $e^{x+y} = e^x e^y$, we separate variables: $\int e^{-y}\,dy = \int e^x\,dx$ and so $-e^{-y} = e^x + C$. The initial condition gives $-1 = 1 + C$ and so $C = -2$. You can write the answer in many ways, for instance $e^x + e^y = 2$.

(b) We have $y' \;=\; t + ty^2 \;=\; t(1+y^2)$. Separating variables:

$$\int \frac{dy}{1+y^2} \;=\; \int t\,dt \quad\text{and so}\quad \arctan y \;==\; t^2/2 + C.$$

4. $r = 2\cos\theta$ and $r = \sqrt{2}$ intersect at $\cos\theta = 1/\sqrt{2}$. Thus $\theta \;=\; \pm\pi/4$. Since $\theta = 0$ gives $2\cos\theta = 2$, $\theta = 0$ is in the region. Thus we integrate from $-\pi/4$ to $\pi/4$:

$$\int_{-\pi/4}^{\pi/4}\Big((2\cos\theta)^2/2 - (2^{1/2})^2/2\Big)\,d\theta \;=\; \int_{-\pi/4}^{\pi/4}(2\cos^2\theta - 1)\,d\theta.$$

5. For $-1 < y < 1$, y increases since $y' > 0$. For $y > 1$, y decreases. Thus $y(t)$ approaches 1 as t gets large. [In (a) in increases with t and in (b) it decreases.]

END OF EXAM