1. (40 pts.) Let $\mathcal{R}$ be the region between the two parabolas $y = x^2$ and $x = y^2$.
Let $\mathcal{V}$ be the volume obtained when $\mathcal{R}$ is rotated about the y-axis.

(a) Sketch the region $\mathcal{R}$. Include in your drawing the coordinates of the point where the parabolas intersect.

Ans. We omit the picture. The region lies in the first quadrant, is bounded below by $y = x^2$ and above by $x = y^2$, and the points of intersection are (0,0) and (1,1). You can find the intersection points from your sketch (and then easily check them in your head). Alternatively, you can find them by solving the equations: Squaring the first and using the second to eliminate y gives $x^4 = x$ and so either $x = 0$ or $x^3 = 1$. The solution to the latter is $x = 1$. From $y = x^2$, we find the corresponding values of y.

(b) The arc length of the boundary of $\mathcal{R}$ that is on the parabola $y = x^2$.

Ans. $\int_0^1 \sqrt{1+4x^2}\,dx$ or $\int_0^1 \sqrt{1+4y^2}\,dy$

(c) The volume of $\mathcal{V}$.

Ans. $\int_0^1 \pi\left(\sqrt{y}^2 - (y^2)^2\right) dy = \int_0^1 \pi(y - y^4)\,dy$

(d) The surface area of $\mathcal{V}$. *Be careful*: $\mathcal{V}$ has what might be called an inner and outer surface. The surface area is the sum of the areas of these two surfaces.

Ans.
$$\int_0^1 2\pi\sqrt{y}\,\sqrt{1+(1/2y^{1/2})^2}\,dy + \int_0^1 2\pi y^2\sqrt{1+(2y)^2}\,dy$$
$$= \int_0^1 2\pi\sqrt{y+1/4}\,dy + \int_0^1 2\pi y^2\sqrt{1+4y^2}\,dy$$

2. (20 pts.) Given the two curves $r = 2$ and $r = 4\cos\theta$ in polar coordinates.

(a) Find the polar coordinates of the points where the curves intersect.

Ans. Setting the two values of r equal gives $\cos\theta = 1/2$ and so $\theta = \pm\pi/3$.

(b) Set up (but *do not evaluate*) an integral for the area that lies inside the curve $r = 4\cos\theta$ but outside the curve $r = 2$; that is, the area of the region for which $2 \le r \le 4\cos\theta$.

Ans. The curve $r = 2$ lies inside $r = 4\cos\theta$ for $-\pi/3 < \theta < \pi/3$ and outside it otherwise. Thus the answer is
$$\int_{-\pi/3}^{\pi/3}\Big((1/2)(4\cos\theta)^2 - (1/2)2^2\Big)\,d\theta = \int_{-\pi/3}^{\pi/3} 2(4\cos^2\theta - 1)\,d\theta,$$
but you need not simplify. Integrating from $5\pi/3$ to $\pi/3$ is NOT correct, but integrating from $5\pi/3$ to $7\pi/3$ is correct. Also, by symmetry, you could have gotten $2\int_0^{\pi/3} 2(4\cos^2\theta - 1)\,d\theta$,

3. (30 pts.) Express each of the following in the form $a + bi$ with a and b real numbers.

(a) $(2+4i)/(1-7i)$

Ans. $\dfrac{2+4i}{1-7i} = \dfrac{(2+4i)(1+7i)}{(1-7i)(1+7i)} = \dfrac{-26+18i}{50} = (-13/25) + (9/25)i = -0.52 + 0.36i$

(b) $\left(2\sqrt{3}+2i\right)^{20}$

Ans. Put $2\sqrt{3}+2i$ in polar form: $r=\sqrt{12+4}=4$ and $\theta=\arctan(1/\sqrt{3})=\pi/6$.
Raise to the 20th power: $4^{20}e^{20\pi i/6}=4^{20}e^{4\pi i/3}=4^{20}\left(-(1/2)-(\sqrt{3}/2)i\right)=-2^{39}-2^{39}\sqrt{3}\,i$

(c) $e^{3+i\pi/2}$

Ans. $e^3(\cos(\pi/2)+i\sin(\pi/2)=0+e^3i=e^3i$

4. (20 pts.) (a) Determine whether $\displaystyle\int_0^\infty e^{-x}dx$ is convergent or divergent.

Ans. $\displaystyle\int_0^\infty e^{-x}dx=\lim_{b\to\infty}\int_0^b e^{-x}dx=\lim_{b\to\infty}\left(-e^{-x}\Big|_0^b=1\right.$, so it is convergent

(b) Use part (a) and the comparison theorem to determine whether $\displaystyle\int_0^\infty \frac{e^{-x}}{2+\sin x}\,dx$ is convergent or divergent.

Ans. Since $2+\sin x\geq 1$, the integrand in (b) is less than or equal to the integrand in (a). Hence we have convergence by the comparison theorem.

5. (30 pts.) Evaluate the following integrals.

(a) $\displaystyle\int \frac{\ln x}{x^2}\,dx$

Ans. Integrate by parts directly or let $\ln x=u$ and integrate by parts.
In the first case, $u=\ln x$ and $dv=x^{-2}\,dx$. Thus $du=dx/x$ and $v=-1/x$ and so
$\displaystyle\int \frac{\ln x}{x^2}\,dx=-(\ln x)/x+\int x^{-2}\,dx=-(\ln x)/x-(1/x)+C.$
In the second case, we have $t=\ln x$ and so $dt=dx/x$ and $x=e^t$. Thus $\displaystyle\int \frac{\ln x}{x^2}\,dx=\int te^{-t}\,dt$ after some algebra. Let $u=t$ and $dv=e^{-t}\,dt$ to integrate by parts. We omit the rest.

(b) $\displaystyle\int \cos x\ \cos(3x)\,dx$

Ans. Using $\cos A\cos B=[\cos(A-B)+\cos(A+B)]/2$ from Section 7.2 and the fact that $\cos(-C)=\cos C$, we have
$\displaystyle\int \cos x\ \cos(3x)\,dx=\frac{1}{2}\int\left(\cos(2x)+\cos(4x)\right)\,dx=\frac{-\sin(2x)}{4}-\frac{\sin(4x)}{8}+C.$

(c) $\displaystyle\int \frac{dx}{x^2\sqrt{1-x^2}}$

Ans. Let $x=\sin t$. Then $dx=\cos t\,dt$ and so

$$\int \frac{dx}{x^2\sqrt{1-x^2}}=\int\frac{\cos t\,dt}{\sin^2 t\ \cos t}=\int \csc^2 t\,dt=-\cot t+C.$$

Drawing the right triangle corresponding to $\sin t=x$, we see that $\cot t=\dfrac{\sqrt{1-x^2}}{x}$ and so our answer is $\dfrac{-\sqrt{1-x^2}}{x}+C$

6. (10 pts.) Write out the partial fraction decomposition of the function $\dfrac{x}{x^2-1}$.

Ans. Since $x^2-1=(x-1)(x+1)$, we have $\dfrac{x}{x^2-1}=\dfrac{A}{x-1}+\dfrac{B}{x+1}=\dfrac{(A+B)x+(A-B)}{x^2-1}$. Thus $A+B=1$ and $A-B=0$. Solving gives $A=B=1/2$ and so $\dfrac{x}{x^2-1}=\dfrac{1/2}{x-1}+\dfrac{1/2}{x+1}$.

7. (15 pts.) Solve the differential equation $\dfrac{dy}{dx}=\dfrac{x^2+1}{x^2y}$ with the initial condition $y(1)=-2$ for y as a *function* of x; that is, find $y(x)$.

Ans. Separating variables gives $\displaystyle\int y\,dy=\int\frac{x^2+1}{x^2}\,dx=\int(1+x^{-2})\,dx$ and so $y^2/2=x-1/x+C$. By the initial condition, $(-2)^2/2=1-1+C$ and so $C=2$. This is not the complete answer since there are two choices for the square root. Solving and using the fact that $y(1)$ is negative, we have $y=-\sqrt{2x-2/x+4}$.

8. (15 pts.) Use Euler's method with step size 1.0 to estimate $x(3.0)$, where $x(t)$ is the solution of the initial value problem

$$\frac{dx}{dt}=x+t,\quad x(0)=0.$$

Ans. We have $h=1$. The following computation gives the answer 4.

n	t_n	x_n	$h(dx/dt)$
0	0	0	0
1	1	0	1
2	2	1	3
3	3	4	

9. (20 pts.) Consider the integral $\displaystyle I=\int_0^2\frac{2}{4x+1}\,dx$.

(a) Use the Midpoint rule with $n=2$ subintervals to approximate I.

Ans. Let $f(x)=2(4x-1)^{-1}$ We have $\Delta x=(2-0)/2=1$. Thus the Midpoint rule gives $1(f(1/2)+f(3/2))=2/(4/2+1)+2/(4\times3/2+1)=2/3+2/7=20/21$.

(b) How large should n be so that the midpoint approximation of I is accurate to within 6×10^{-4}?

Ans. We need to bound $|f''(x)|$ for $0\le x\le 2$. We have $f'(x)=-8(4x+1)^{-2}$ and $f''(x)=64(4x+1)^{-3}$. Since $1\le 4x+1\le 5$, we have $64\ge f''(x)\ge 64/5^3$. Thus we can take $K=64$ in the formula $E_M\le K(b-a)^3/24n^2$. This gives $E_M\le 64\times2^3/24n^2$ and so we need $64\times8/24n^2\le 6\times10^{-4}$. This gives $n^2\ge\dfrac{64\times8\times10^4}{24\times6}=\dfrac{32\times10^4}{9}$. Hence $n\ge\sqrt{\dfrac{32\times10^4}{9}}=\dfrac{400}{\sqrt{3}}$.