1. (80 pts.) Evaluate the following integrals. Remember to show your work!

   (a) [7.5 #9] The answer is $1 - \sqrt{3}/2$. It can be done as an indefinite integral or by carrying the limits along. There are two reasonable substitutions: $x = \sin t$ and $1 - x^2 = u$. The first, without limits, using $dx = \cos t\, dt$:

$$\int \frac{x}{\sqrt{1-x^2}}\, dx = \int \frac{\sin t}{\cos t}(\cos t\, dt) = \int \sin t\, dt = -\cos t + C = -\sqrt{1-x^2} + C.$$

   Had the limits been carried along, $\int_0^{1/2}$ over x would become $\int_0^{\pi/6}$ over t. The second substitution in the definite integral form, using $-2x\, dx = du$:

$$\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\, dx = \int_1^{3/4} \frac{-u^{-1/2}}{2}\, du = -u^{1/2}\Big|_1^{3/4} = 1 - \sqrt{3/4}.$$

   (b) [7.5 #22] Multiply out and integrate:

$$\int \sqrt{t}\left(t + \sqrt[3]{t}\right)\, dt = \int (t^{3/2} + t^{5/6})\, dt = 2t^{5/2}/5 + 6t^{11/6}/11 + C.$$

   (c) Replace $\cos^2 t$ with $1 - \sin^2 t$ and let $\sin t = u$:

$$\int \sin^2 t \cos^3 t\, dt = \int \sin^2 t(1 - \sin^2 t)\cos t\, dt = \int u^2(1-u^2)\, du$$
$$= u^3/3 - u^5/5 + C = \frac{\sin^3 t}{3} - \frac{\sin^5 t}{5} + C.$$

   (d) [7.1 #9] Either integrate by parts twice, or first let $\ln x = t$ and then integrate by parts twice In the first case, we are always have $dv = dx$ and in the latter case, $dv = e^t\, dt$.

$$\int (\ln x)^2\, dx = x(\ln x)^2 - 2\int x\frac{\ln x}{x}\, dx = x(\ln x)^2 - 2x\ln x + 2\int x\frac{1}{x}\, dx$$
$$= x(\ln x)^2 - 2x\ln x + 2x + C.$$

$$\int (\ln x)^2\, dx = \int t^2 e^t\, dt = t^2e^t - 2\int te^t\, dt = t^2e^t - 2te^t + 2\int e^t\, dt$$
$$= t^2e^t - 2te^t + 2e^t + C = (\ln x)^2x - 2(\ln x)x + 2x + C.$$

   (e) The answer is *not* zero. This is an improper integral because the integrand blows up at $x = 0$ and it *diverges*. To see this, we split the integral into two:

$$\int_{-1}^{1} \frac{1}{x^3}\, dx = \int_{-1}^{0} \frac{1}{x^3}\, dx + \int_0^1 \frac{1}{x^3}\, dx.$$

Neither integral converges. You may have learned that $\int_0^b x^{-p}\,dx$ converges if and only if $p < 1$ and can quote that fact. If not, you can look at either of the integrals. We do one:

$$\int_0^1 \frac{1}{x^3}\,dx = \lim_{a\to 0^+} \int_a^1 x^{-3}\,dx = \lim_{a\to 0^+} (a^{-2} - 1)/2 = +\infty.$$

The answer infinity is *not acceptable* because the integral from -1 to 0 diverges to $-\infty$ and so you would have the "indeterminate form $\infty - \infty$. (If you forgot the meaning of that term, see l'Hospital's rule section.)

2. (15 pts.)[6.2 #33] Set up the integral for the volume of the solid obtained by rotating the region bounded by

$$y = 0, \quad y = \sin x, \quad 0 \le x \le \pi$$

about the line $y = 1$.

Your answer should include a sketch of the region together with the line about which the region is being rotated.

Ans. We omit the sketch. It is one "hump" of the sine curve bounded below by the x-axis. The integral is

$$\int_0^\pi \pi\Big(1^2 - (1-\sin x)^2\Big)\,dx = \pi \int_0^\pi (2\sin x - \sin^2 x)\,dx.$$

(You need not expand the first integrand as was done here.)

3. (30 pts.) I have a function $f(x)$ and know that

$$\begin{aligned} +4 \le f'(x) \le +20 \quad &\text{for } 0 \le x \le 5 \quad \text{and} \\ -3 \le f''(x) \le -2 \quad &\text{for } 0 \le x \le 5. \end{aligned}$$

I want to use either the Midpoint rule or the Trapezoidal rule to obtain a lower bound for $\int_0^5 f(x)\,dx$; that is, an estimate which is smaller than $\int_0^5 f(x)\,dx$.

(a) Which should I use (Midpoint or Trapezoidal) and <u>*why*</u>?
Hint: A sketch of $f(x)$ may help you find the answer.

Ans. For a concave down function, the Midpoint overestimates and the Trapezoidal underestimates, so we *use the Trapezoidal rule.*

(b) I would like guarantee that the error in my estimate is no larger than $0.001 = 10^{-3}$. How large must I make n to guarantee this?

Ans. The error bound for the Trapezoidal rule is $K(b-a)^3/12n^2$, where K is such that $|f''(x)| \le K$. We have $a = 0$, $b = 5$, and can take $K = 3$. Thus we want $3 \times 5^3/12n^2 < 10^{-3}$, which gives us

$$n \ge \sqrt{\frac{3\times 5^3}{12\times 10^{-3}}} = \sqrt{2\times 5^6} = 125\sqrt{2}.$$

You need not have simplified your answer. If you made the mistake of choosing the Midpoint rule for (a) and then computed n correctly for that choice, you will receive full credit on (b). The Midpoint error bound is $K(b-a)^3/24n^2$ and it gives the answer $n \ge 125$.