1. (a) Since this is of the form 0/0, we apply l'Hôpital's rule:

$$\lim_{x\to 0}\frac{e^x-1}{\sin x}=\lim_{x\to 0}\frac{e^x}{\cos x}=\frac{1}{1}=1.$$

(b) You could convert to polar form: $r=\sqrt{2}$ and $\theta=\pi/4$, giving $r^{30}=2^{15}$ and $30\theta=15\pi/2=8\pi-\pi/2$. Thus $(1+i)^{30}=-2^{15}i$. Alternatively, you can take powers:

$$(1+i)^{30}=((1+i)^2)^{15}=(2i)^{15}=2^{15}i^{15}=2^{15}i^3=-2^{15}i.$$

2. (a) The denominator factors as $x(x^2-1)=x(x-1)(x+1)$. Hence

$$\frac{2}{x^3-x}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}=\frac{A(x^2-1)+B(x^2+x)+C(x^2-x)}{x(x-1)(x+1)}.$$

Thus $(A+B+C)x^2+(B-C)x-A=2$. From constant and linear terms, $A=-2$ and $B=C$. Thus the quadratic gives $-2+2B=0$ and so $B=1$. Thus we have

$$\int\frac{2}{x^3-x}\,dx=\int\left(\frac{-2}{x}+\frac{1}{x-1}+\frac{1}{x+1}\right)dx=\ln\left|\frac{x^2-1}{x^2}\right|+C=\ln|1-x^{-2}|+C.$$

(b) First method: Integrate by parts with $u=\ln x$ and $dv=dx$. This gives

$$\int_1^e \ln x\,dx=x\ln x\Big|_1^e-\int_1^e x\frac{dx}{x}=e-\left(x\Big|_1^e\right.=e-(e-1)=1.$$

Second method: Change variables with $\ln x=t$ and so $dx/x=dt$. Thus $dx=x\,dt=e^t\,dt$. Substituting and integrating by parts:

$$\int_1^e \ln x\,dx=\int_0^1 te^t\,dt=te^t\Big|_0^1-\int_0^1 e^t\,dt=e-(e-1)=1.$$

(c) Replace $\cos^2 x$ with $1-\sin^2 x$ and use the fact* $\int\sin^{n-1}x\cos x\,dx=(\sin^n x)/n+C$ to get

$$\int\sin^4 x\cos^3 x\,dx=\int\sin^4 x\cos x\,dx-\int\sin^6 x\cos x\,dx=\frac{\sin^5 x}{5}-\frac{\sin^7 x}{7}+C.$$

(d) You may recognize the integrand as the derivative of the inverse secant as so write down the answer $\sec^{-1}x+C$ immediately. Otherwise, use the trig substitution $t=\sec x$ with $dt=\sec x\ \tan x\,dx$:

$$\int\frac{1}{t\sqrt{t^2-1}}\,dt=\int\frac{\sec x\ \tan x\,dx}{\sec x\ \tan x}=\int dx=x+C=\sec^{-1}t+C.$$

* You can prove the fact with the substitution $t=\sin x$.

3. To find the cube roots of $1+i$, we convert it to polar form: $r=\sqrt{2}=2^{1/2}$ and $\theta=\pi/4$. The roots all have $r=2^{1/6}$. The values of θ are $\pi/12$, $2\pi/3+\pi/12=3\pi/4$ and $4\pi/3+\pi/12=17\pi/12$. The angle $17\pi/12$ is the same as the angle $-7\pi/12$, so you can use either. Since the problem did not specify, you could leave the answer in polar form and receive full credit. Also, you could write the answers as

$$2^{1/6}\cos(\pi/12)+i2^{1/6}\sin(\pi/12),$$
$$2^{1/6}\cos(\pi/4)+i2^{1/6}\sin(\pi/4),$$
$$2^{1/6}\cos(17\pi/12)+i2^{1/6}\sin(17\pi/12).$$

4. Since the region is bounded by the y-axis and $x=8$, all integrals go from $x=0$ to $x=8$. Since $y=(4x+4)^{1/2}$, $dy/dx=2(4x+4)^{-1/2}$. The answers are

$$\begin{aligned}\text{volume} &= \int_{x=0}^{x=8}\pi y^2\,dx=\pi\int_0^8(4x+4)dx\\ \text{area} &= \int_{x=0}^{x=8}2\pi y\sqrt{1+(y')^2}\,dx=2\pi\int_0^8(4x+4)^{1/2}\sqrt{1+4(4x+4)^{-1}}\,dx\\ &= 2\pi\int_0^8\sqrt{(4x+4)+4}\,dx.\end{aligned}$$

If you wished, you could add in the two circular areas at the ends, namely $\pi 4^2$ and $\pi(36)^2$.

5. Separate variables and integrate. There cannot be division by zero since an exponential is never zero. We have

$$\int x\,dx=\int te^t\,dt=(t-1)e^t+C,$$

using integration by parts. The left side is $x^2/2$. Using $x(0)=1$ gives us $1/2=-1+C$ and so $C=3/2$. Finally we multiply the solution by two and take the *positive* square root (since $x(0)=1$ is positive:

$$x=\sqrt{2(t-1)e^t+3}.$$

6. (a) With $f(x)=1/x$, we have $f''(x)=3/x^3>0$ and so the function is concave up. For functions which are concave up, the Trapezoidal rule overestimates and the Midpoint rule underestimates. If you wish to show what happens, you can draw a picture.

   (b) We know that the error in the Trapezoidal and Midpoint rule is roughly of the form C/n^2, where C depends on the integrand and the rule. Thus $E\approx C/(10)^2$ and we want $C/(20)^2=(C/(10^2))/4\approx E/4$. Thus the answer is $E/4$.

   Instead, you could use the observation in the Section 7.7 where $\int dx/x$ is discussed: doubling n reduces the error by a factor of about four.

7. I'll omit the sketch. It looks like a circle with the bottom stretched and the top pushed in. The area is $\frac{1}{2}\int_0^{2\pi}(4-\sin\theta)^2\,d\theta$.

8. (a) The information says that $-y' = K\,y^{1/2}$ and the initial condition is $y(0) = 25$. We are also told that $y(2) = 16$.

   (b) Separate variables and integrate: $\int y^{-1/2}\,dy = \int -K\,dt$ and so $2y^{1/2} = -Kt + C$. (*Warning*: we must be careful about division if $y = 0$.)
   At $t = 0$ we have $2 \times 5 = C$, so $C = 10$. Thus

$$2y^{1/2} = 10 - Kt, \text{ and we have not used } y(2) = 16.$$

   Using the last piece of information gives $8 = 10 - 2K$ and so $K = 1$. Solving for y we get $y = (5 - t/2)^2$.

   (c) With $t = 10$, we get $y = 0$; that is, the tank is empty. The tank stays empty since no water is flowing in. Hence $y = 0$ for all $t \geq 10$, including $t = 20$.
   If we blindly plug $t = 20$ into the formula, we get $y = 25$, which is absurd! The problem arises because of division by zero when $t = 10$. (See the warning in the solution to (b)).