1. (80 pts.) Evaluate the following integrals. Remember to show your work!

   (a) [7.5, #6] $\displaystyle\int (\sin x)\,(\cos(\cos x))\,dx$

   Ans. Use $u = \cos x$ to get rid of the messy cosine stuff. Then $du = -\sin x\;dx$ and so we have $\int -\cos u\;du = -\sin u + C$, which gives $-\sin(\cos x) + C$.

   (b) [7.5, #10] $\displaystyle\int \frac{t^2}{\sqrt{1-t^2}}\,dt$

   Ans. Set $t = \sin\theta$. Then $dt = \cos\theta\;d\theta$ and the integral becomes $\int \sin^2\theta\;d\theta = (1/2)\int(1-\cos 2\theta)d\theta$, which is $(\theta/2) - (\sin 2\theta)/4 + C$. You now must substitute back in for θ:
   $$\frac{\sin^{-1} t}{2} + \frac{\sin(2\sin^{-1} t)}{4} + C.$$
   You can leave your answer in this form; however, you could simplify it using $\sin(2\theta) = 2\sin\theta\;\cos\theta = 2t\sqrt{1-t^2}$.

   (c) [7.5, #26] $\displaystyle\int \sin(t^{1/2})\;dt$

   Ans. Set $t^{1/2} = x$ to obtain $dt = 2x\;dx$ and $\int 2x\sin x\;dx$. This integral can be done by parts with $x = u$ and $\sin x\;dx = dv$. We obtain $-2x\cos x + 2\int\cos x\;dx$. Integrating and substituting back gives $-2t^{1/2}\cos t^{1/2} + 2\sin t^{1/2} + C$.

   (d) [7.5, #44] $\displaystyle\int \frac{1+e^x}{1-e^x}\,dx$

   Ans. The are various possibilities. Setting $e^x = u$, we have $dx = du/e^u$ and the integral becomes
   $$\int \frac{1+u}{1-u}\,\frac{du}{u} = \int\left(\frac{2}{1-u} + \frac{1}{u}\right)du,$$
   by partial fractions. Integrating and substituting back gives $-2\ln|1-e^x|+x+C$.

2. (20 pts.)[see 7.7, #2] Let $f(x) = e^{-x^2/2}$ and let $I = \int_0^1 f(x)\;dx$.
   It can be shown that $f'(x) < 0$ for $x > 0$ and $f''(x) < 0$ for $|x| \le 1$.

   The left, right, Trapezoidal, and Midpoint Rules were used to estimate I and the same number of subintervals were used in each case. Call the estimates L, R, T, and M, respectively. Order I, L, M, R, and T from smallest to largest.

Ans. The function is concave and decreasing by the second and first derivative information, respectively. Since the function is decreasing, L is bigger than everything else and R is smaller than all others. Since the function is concave, $T < I < M$. Hence $R < T < I < M < L$.

You could draw a picture, showing that the estimate from x_{i-1} to x_i involves rectangles for L and R and trapezoids for T and M. (The trapezoid for M is obtained by drawing a tangent to the curve at $(x_{i-1} + x_i)/2$. The previous comments about the order are obvious from the picture.

3. (25 pts.) Determine which of the following integrals are divergent and which are not. Evaluate all integrals which are NOT divergent.

(a) $\displaystyle\int_0^1 \frac{2x}{x^2-4x+3}\,dx$ (b) $\displaystyle\int_2^4 \frac{2x}{x^2-4x+3}\,dx$ (c) $\displaystyle\int_4^6 \frac{2x}{x^2-4x+3}\,dx$

Note that $x^2-4x+3=(x-1)(x-3)$.

Ans. The denominator vanishes at $x=1$ and $x=3$ and nowhere else. By what I posted on the course web page, the first two integrals diverge; however, I won't use that. Using partial fractions,

$$\frac{2x}{(x-1)(x-3)} = \frac{-1}{x-1} + \frac{3}{x-3}$$

and so the indefinite integral is $-\ln|x-1|+3\ln|x-3|+C$. This blows up as $x\to 1$ or $x\to 3$, so the first two integrals diverge. The third integral is

$$(-\ln 5 + 3\ln 3) - (-\ln 3 + 3\ln 1) = -\ln 5 + 4\ln 3.$$

You can leave the answer this way or write it as $\ln(81/5)$.