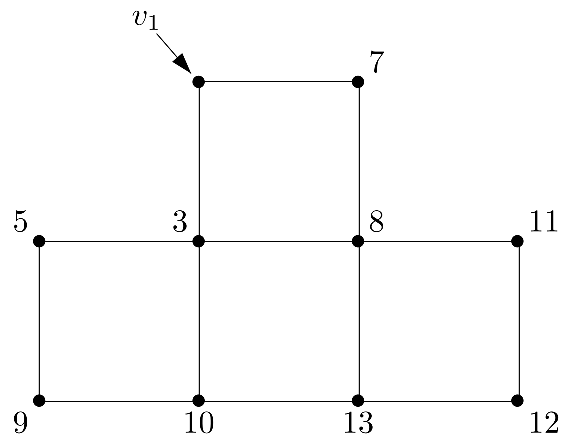
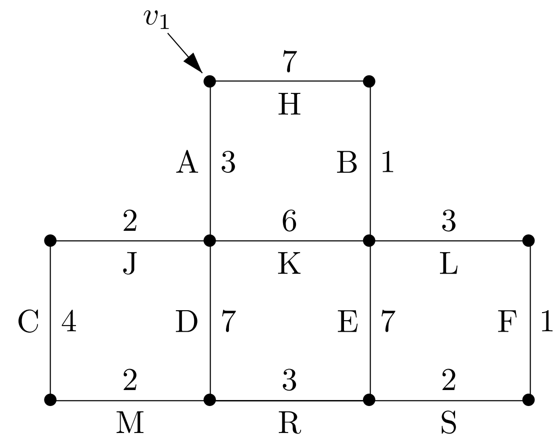
1. (30 pts.) Recall that Dijkstra's algorithm finds shortest paths from $v_1$ to all other vertices by adding edges linking in the closest points. In the graph shown below, each edge is bidirectional; that is, you can travel in either direction on it. **Edges are labeled with upper case letters**. (Two copies of the graph are provided so you can use one as a "worksheet" if you wish.)

   (a) List edges in order chosen by algorithm: A J H B C D L F R

   (b) At each vertex, give the length of the shortest path from $v_1$ to the vertex. *Indicate which graph has your answer.*




2. (25 pts.) Consider the following eight complexity categories (remember $\lg = \log_2$):

$$\Theta(n) \quad \Theta(n^2) \quad \Theta(2^n) \quad \Theta(3^{\lg n}) \quad \Theta(n^{\lg n}) \quad \Theta(n \lg n) \quad \Theta((\sqrt{n} + \ln n)^2) \quad \Theta(2^{n+\lg n}).$$

   (a) Which are equal?

$$\Theta(n) = \Theta((\sqrt{n} + \ln n)^2)$$

   (b) Arrange the distinct classes in order from slowest growing to fastest growing. In other words, if $\Theta(f(n))$ is to the left of $\Theta(g(n))$, then $f(n) \in o(g(n))$.

$$\Theta(n) \quad \Theta(n \lg n) \quad \Theta(3^{\lg n}) \quad \Theta(n^2) \quad \Theta(n^{\lg n}) \quad \Theta(2^n) \quad \Theta(2^{n+\lg n}).$$

3. (30 pts.) The average running time for an algorithm is a nondecreasing function of $n$ and satisifies $T(4n) = T(2n) + 2T(n)$ for all $n > 0$. Furthermore, $T(1) = 1$ and $T(2) = 3$.

   (a) Determine $T(2^k)$ as a function of the integer $k$.
   *Hint*: Set $t_k = T(2^k)$.

   Ans. By the hint, $t_{k+2} = t_{k+1} + 2t_k$, where $t_0 = 1$ and $t_1 = 3$. Since the roots of $x^2 = x + 2$ are $x = -1$ and $x = 2$, the general solution to the recursion is

$$t_k = A(-1)^k + B2^k.$$

   With $k = 0$ 1, we have $A + B = 1$ and $-A + 2B = 3$. Hence $B = 4/3$ and $A = -1/3$. Thus $T(2^k) = (2^{k+2} - (-1)^k)/3$.

   (b) Determine the complexity class of $T(n)$.

   Ans. $T(n) \in \Theta(n)$ by Theorem B.4.

4. (30 pts.) Suppose we have two sorted lists $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, both of length $n$, that we want to merge to obtain a sorted list of length $2n$, say $c_1, \ldots, c_{2n}$. To do this, we must decide where the $a_i$'s fit among the $b_j$'s to produce the $c$ list. The number of choices for this is $\binom{2n}{n} \geq 4^n/(2\,n^{1/2})$.

   Suppose the merge is done comparisons of keys. Using the above information, derive a lower bound for the worst case number of key comparisons that are needed. Explain your reasoning; don't just give an answer.

   Ans. Each comparison allows us the split the possibilities into two parts. The decision tree will be binary and must have at least $\binom{2n}{n}$ leaves. Since the longest from root to leaf in such a tree is at least the log base 2 of the number of leaves, $W(n) \geq \lceil \lg \binom{2n}{n} \rceil$. You could leave off the ceiling function. You could also use the lower bound for the binomial coefficient to get

$$W(n) \geq 2n - \lg 2 - (\lg n)/2.$$

   By the way, this is nearly achieved by the merge process in mergesort: It's worst case number of comparisons is $2n - 1$.

5. (30 pts.) Here is an informal description of a routine `Proc` that is looking for $x$ in a sorted list $S$. The parameters are the ends of the list. While it is looking it does some processing in `ProcLow` and `ProcHigh`.

```
Proc(lo,hi)
    If lo > hi we are done.
    k = ⌊(lo + hi)/2⌋.
    If S[k] = x, we are done.
    If S[k] < x
        Call ProcHigh(k,hi) and Proc(k + 1,hi)
    Else
        Call ProcLow(lo,k) and Proc(lo,k − 1)
    Endif.
End
```

We begin by calling `Proc(1,`$n$`)`. Most of the time is spent in `ProcLow` and `ProcHigh`. In fact, `ProcLow(`$a$`,`$b$`)` requires $\lg(b-a+1)$ basic operations and `ProcHigh(`$a$`,`$b$`)` requires $(b-a+1)$ basic operations. (You do *not* need to know what any of this code is supposed to do.)

(a) Let $W(n)$ be the worst case running time for `Proc(1,`$n$`)`. Give a recursion and initial condition for $W(2^n)$. (In the worst case, $x$ is not in the list.)

Ans. When the length of the list is even, the part above $k$ is exactly half of the list. The part below $k$ is one shorter and also requires less processing time because of the "lg". Hence the worst case will be to always take the right half. Thus $W(n) = W(n/2) + n/2$. When $n = 2^k$, $W(2^k) = W(2^{k-1}) + 2^{k-1}$.

(b) Let $A(n)$ be the average running time for `Proc(1, `$n$`)`. Assuming $x$ is not in the list and the probability that $S[k] < x$ is $1/2$, give a recursion for $A(n)$. You need *not* give an initial condition.

Ans. When $n$ is even, the reasoning in the previous answer gives

$$A(n) = \frac{A(n/2) + n/2}{2} + \frac{A(n/2-1) + \lg(n/2-1)}{2}.$$

When $n$ is odd, similar reasoning gives

$$A(n) = \frac{A((n-1)/2) + (n-1)/2}{2} + \frac{A((n-1)/2) + \lg((n-1)/2)}{2}.$$

There's no need to write this as a single recursion, but you can. One way to do so is

$$A(n) = \frac{A(\lceil (n-1)/2\rceil) + \lceil (n-1)/2\rceil}{2} + \frac{A(\lfloor (n-1)/2\rfloor) + \lg(\lfloor (n-1)/2\rfloor)}{2}.$$

6. (65 pts.) Indicate whether true or false. Beware of guessing:

correct answer +5pts. incorrect answer −3pts. no answer 0pts

T $\Theta(2^{n+2}) = \Theta(2^n)$.

T $\Theta((n+2)^2) = \Theta(n^2)$.

F $\Theta(2^{n+\lg n}) = \Theta(2^n)$.

T $\Theta((n+\lg n)^2) = \Theta(n^2)$.

T Greedy algorithms are called "greedy" because they make the best choice at the present time, without concern for the future.

T Dynamic programming algorithms use a bottom up approach.

F Divide and conquer algorithms use a bottom up approach.

T If a divide an conquer algorithm requires recomputing the same quantity many times, it is a good idea to look for a dynamic programming algorithm.

T No greedy algorithm is known for the 0-1 Knapsack Problem.

F It is usually fairly easy to determine average and worst-case time complexities for backtracking algorithms.

F There is a search algorithm that uses comparison of keys and is significantly faster on average and in the worst case than binary search.

F There is a sorting algorithm that uses comparison of keys and is significantly faster on average and in the worst case than mergesort.

T Quicksort has a good average run time and a poor worst-case run time.