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1. Find a_n when $a_0 = 0$ and $a_{n+1} = 3a_n + 1$ when $n \ge 0$.
Answer: Let $A(x) = \sum_{n\ge 0} a_n x^n$, the generating function for the a_ns. Multiply both sides of the recursion by x^{n+1} and sum:

$$A(x) - a_0 = 3xA(x) + \sum_{n=0}^{\infty} x^{n+1} = 3A(x) + \frac{x}{1-x}.$$

Hence

$$A(x) = \frac{x}{(1-x)(1-3x)} = \frac{1/2}{1-3x} - \frac{1/2}{1-x} = \sum (3^n/2)x^n - \sum (1/2)x^n.$$

Thus $a_n = (3^n - 1)/2$.

2. For the graph shown here, find *two different* lineal spanning trees with A as root vertex
Answer: There six possible trees. For all of them, A has three principal subtrees, each of which is a path. One tree consists of F alone. Another consists of EG or GE. The third consists of CBD, CDB, DBC, or DCB.

3. (a) A strictly decreasing function from $\underline{k}$ to $\underline{n}$ is given by $f(i) = k+2-i$ for $1 \le i \le k$. Prove that its rank is k. (n is much larger than k.)
Answer: Recall that $\mathrm{RANK}(f) = \displaystyle\sum_{i=1}^{k} \binom{f(i)-1}{k+1-i}$. Since the binomial coefficients are $\binom{k+2-i-1}{k+1-i} = \binom{k+1-i}{k+1-i} = 1$, the sum is k.
(b) A strictly decreasing function from $\underline{k}$ to $\underline{n}$ is given by $f(i) = k+3-i$ for $1 \le i \le k$. Prove that its rank is $\dfrac{k(k+3)}{2}$.
Answer: As in (a), but the binomial coefficients are now $\binom{k+2-i}{k+1-i} = k+2-i$. Hence the terms in the sum for the rank go from $k+1$ down to 2. We use induction on k to prove that the sum is the given answer. When $k = 1$, the result is true. For k, the formula and the induction hypothesis give

$$\text{Sum} = (k+1) + \frac{(k-1)((k-1)+3)}{2} = \frac{k(k+3)}{2}.$$

4. If T is an unlabeled binary RP-tree with more than one leaf, let T_1 and T_2 be its left and right principal subtrees and let $|T|$ be the number of leaves in T. Using Exercise 9.1.12(c) or otherwise, prove: (You do *NOT* need to prove Exercise 9.1.12.)

   When $n \ge 2$ over half the unlabeled binary RP-trees with n leaves have either $|T_1| = 1$ or $|T_2| = 1$.

Answer: When $n = 2$, the result is true. When $n > 2$, the number of trees with

$|T_1| = 1$ is $b_1 b_{n-1} = b_{n-1}$ and likewise for $|T_2| = 1$. The the fraction of trees with either $|T_1| = 1$ or $|T_2| = 1$ is $2b_{n-1}/b_n$. From the exercise, $(n+1)b_{n+1} = 2(2n-1)b_n$ and so

$$\frac{2b_{n-1}}{b_n} = \frac{2nb_{n-1}}{nb_n} = \frac{2nb_{n-1}}{2(2(n-1)-1)b_{n-1}} = \frac{n}{2n-3} = \frac{1}{2-3/n} > \frac{1}{2}.$$

5. An *oriented* simple graph is a simple graph which has been converted to a digraph by assigning an orientation to each edge.
   Answer: This problem appeared on the first hour exam.
   (a) Prove that the number of n-vertex oriented simple graphs is $3^{\binom{n}{2}}$.
   (b) State and prove a formula for the number of n-vertex oriented simple graphs that have exactly q edges.

6. Let B_n be the number of partitions of $\underline{n}$. It was proved in the text that $B_{n+1} = \sum_{k=0}^{\infty} \binom{n}{k} B_k$ for $n \ge 0$ provided $B_0 = 1$. Although the sum is infinite, only finitely many terms are nonzero since $\binom{n}{k} = 0$ if $k > n$. Prove by induction that

$$B_n = \frac{1}{e} \sum_{i=0}^{\infty} \frac{i^n}{i!} \quad \text{for} \quad n \ge 0, \qquad \text{where } 0^0 \text{ equals } 1.$$

Hints: Remember that $\sum_{i=0}^{\infty} \sum_{k=0}^{\infty}(\text{anything}) = \sum_{k=0}^{\infty} \sum_{i=0}^{\infty}(\text{same thing})$.
You may find $(j+1)^m/j! = (j+1)^{m+1}/(j+1)!$ useful in your proof.
Answer: The case $n = 0$ is straightforward:

$$\sum_{i=0}^{\infty} \frac{i^0}{i!} = \sum_{i=0}^{\infty} \frac{1}{i!} = e.$$

For induction, we use the recursion for B_{n+1} and the formula for B_k:

$$\begin{aligned} B_{n+1} &= \sum_{k=0}^{\infty} \binom{n}{k} B_k = \sum_{k=0}^{\infty} \binom{n}{k} \frac{1}{e} \sum_{i=0}^{\infty} \frac{i^k}{i!} \\ &= \frac{1}{e} \sum_{i=0}^{\infty} \frac{1}{i!} \sum_{i=0}^{\infty} \binom{n}{k} i^k \\ &= \frac{1}{e} \sum_{i=0}^{\infty} \frac{1}{i!} (1+i)^n, \end{aligned}$$

where the last equality is due to the binomial theorem. We want this to equal

$$\frac{1}{e} \sum_{i=0}^{\infty} \frac{i^{n+1}}{i!}.$$

Using the hint and the formula we derived so far,

$$B_{n+1} = \frac{1}{e}\sum_{i=0}^{\infty}\frac{(1+i)^{n+1}}{(i+1)!} = \frac{1}{e}\sum_{j=1}^{\infty}\frac{j^{n+1}}{j!}.$$

Since $0^{n+1} = 0$ for $n \geq 0$, we may extend to the last sum to include the term $j = 0$ because that term will equal zero.

7. Two 52-card decks are combined to form a deck in which there are two identical copies of each card. The number of possible 2-card hands is $\binom{52}{2} + 52$ because you can choose 2 distinct cards OR choose two cards the same.
Answer: This could be done in more than one way. I'll give one. If both copies of a card are in the hand, call the card a "repeater."

   (a) Prove that the number of 3-card hands is $\binom{52}{3} + 52 \times 51$.
   Answer: (Choose 3 distinct cards) OR (choose a distinct card AND a repeater). The first part is $\binom{52}{3}$. The second part is 52×51.

   (b) Calculate the number of 4-card hands.
   Answer: (Choose 4 distinct cards) OR (choose 2 repeaters) OR (choose 2 distinct cards AND a repeater). The first can be done in $\binom{52}{4}$ ways. The second can be done in $\binom{52}{2}$ ways. The last can be done in $\binom{52}{2} \times 50 = \dfrac{52 \times 51 \times 50}{2}$ ways. Thus we have

$$\binom{52}{4} + \binom{52}{2} + \frac{52 \times 51 \times 50}{2}.$$

8. Let q_n be the number of partitions of n all of whose parts are odd and let d_n be the number of partitions of n all of whose parts are distinct. Let $q_0 = d_0 = 1$. For example, $q_7 = d_7 = 5$ because

$$7 = 1+1+5 = 1+1+1+1+3 = 1+3+3 = 1+1+1+1+1+1+1$$

and

$$7 = 1+6 = 2+5 = 1+2+4 = 3+4.$$

   (a) Let $D(x) = \sum_{n=0}^{\infty} d_n x^n$. Prove that $D(x) = \prod_{k=1}^{\infty}(1 + x^k)$.
   Answer: We have the Rules of Sum and Product for partitions. (See the second half of Example 11.14.)

   (Choose the number of 1s) AND (choose the number of 2s), etc.

   For choosing the number of ks, there can be 0 OR 1 and so we have $(x^k)^0 + (x^k)^1 = 1 + x^k$. Multiplying because of AND, gives the stated formula.

(b) State and prove a formula for $Q(x) = \sum_{n=0}^{\infty} q_n x^n$.

Answer: If k is odd, the kth part can appear any number of times, so we get $1 + x^k + (x^k)^2 + \cdots = (1 - x^k)^{-1}$. Thus

$$Q(x) = \prod_{k \text{ odd}} \frac{1}{1 - x^k}.$$

(c) Prove that $Q(x) = D(x)$ and hence $q_n = d_n$ as follows: Multiply $D(x)$ by $F(x) = \prod_{k=1}^{\infty} \left(1 - x^k\right)$ and combine the two factors having the same power of x. Divide the result by $F(x)$ and show that this is the same as the $Q(x)$ you obtained in (b).

Answer: Proceeding as instructed:

$$D(x)F(x) = \left(\prod_{k=1}^{\infty}(1 + x^k)\right)\left(\prod_{k=1}^{\infty}(1 - x^k)\right) = \prod_{k=1}^{\infty}(1 - x^{2k}) = \prod_{k \text{ even}} (1 - x^k)$$

and so

$$\frac{D(x)F(x)}{F(x)} = \frac{\prod_{k \text{ even}} (1 - x^k)}{\prod_{\text{all } k} (1 - x^k)} = \frac{1}{\prod_{k \text{ odd}} (1 - x^k)}.$$