1. AUTOMATIC contains 2 pairs of repeated letters and 7 distinct letters for a total of 9 letters. For $4 \le n \le 7$, find a formula for the number of n-letter "words" that can be made from this collection of letters. Your answer should be a formula involving n, not 4 separate numbers for $n = 4,\ 5,\ 6,\ 7$.

2. Let $P_k(n)$ be the number of permutations of the set $\{1, \ldots, n\}$ having no cycles of length greater than k. Thus $P_1(n) = 1$ for $n > 0$ since all cycles are of length 1. For later convenience, define

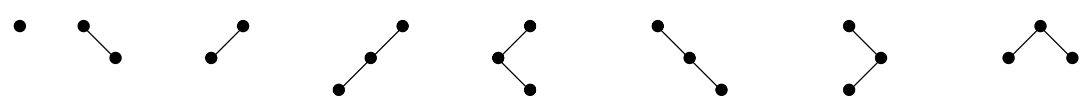
$$P_k(n) = \begin{cases} 0, & \text{if } n < 0 \\ 1, & \text{if } n = 0. \end{cases}$$

   (a) By considering the cycle containing $n+1$, prove that $P_2(n+1) = P_2(n) + nP_2(n-1)$ for $n \ge 0$. (Be careful for small values of n.)

   (b) State and prove a similar recursion for $P_3(n+1)$.

3. Define a "special" tree to be a rooted plane tree which is either
   - a single vertex (the root) or
   - a root vertex that is joined to either a left tree or a right tree or both a left and right tree.

   The special trees with at most 3 vertices are



   Let s_n be the number of n-vertex special trees and let $S(x) = \sum_{n=1}^{\infty} s_n x^n$. By the picture, $s_0 = 0$, $s_1 = 1$, $s_2 = 2$, and $s_3 = 5$.

   (a) Use the definition of special trees to obtain an equation that can be solved for $S(x)$.

   (b) Prove that the solution to the equation you obtained in (a) is

$$S(x) = \frac{1 - 2x - \sqrt{1 - 4x}}{2x}.$$

   (c) As in the text, let b_n be the number of n-leaf binary RP-trees. Prove that $s_n = b_{n+1}$ for $n > 0$. (In case you've forgotten, the generating function for the b_n's satisfies $B(x) = x + B(x)^2$.)

4. In this problem, you look at sequences made from $\{0, \ldots, k\}$ where none of $1, \ldots, k$ may be adjacent to itself. (We also allow the empty sequence.) Call such a sequence "special." For example, 011 is not special, but 012 and 100 are special. Call special sequences without zeros "zero-free."
Notice (you need not prove this) that a special sequence is either
   - a zero-free special sequence or
   - a special sequence, followed by a 0, followed by a zero-free special sequence.

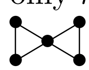
   Let s_n be the number of n-long special sequences and let z_n be the number of n-long zero-free special sequences. Let $S(x)$ and $Z(x)$ be their generating functions.

   (a) Prove that $S(x) = Z(x) + S(x)xZ(x)$.

   (b) Prove that $z_0 = 1$ and $z_n = k(k-1)^{n-1}$ for $n \geq 1$.

   (c) Express $S(x)$ as a rational function of x; that is, a ratio of two polynomials in x.

5. (a) Prove that a simple connected graph with v vertices and $v + n$ edges has at least $n + 1$ cycles for $n \geq 0$. (See the end of the exam for facts you may find useful.)

   (b) For every $n \geq 0$, construct a simple connected graph that has v vertices and $v + n$ edges for some v, and has only $n + 1$ cycles.
   Hint: For $n = 1$, consider 

For a simple connected graph with more than one vertex, the following are equivalent:
- It is a tree.
- It has no cycles.
- For every pair of points $u \neq v$, there is a unique path from u to v.
- The number of vertices is one more than the number of edges.

You may use any of these without proof.