Each problem is worth 12 points. Please start each problem on a new page.

1. Find a minimum weight spanning tree for the graph shown below. Vertices are indicated by letters and weights of edges are given numerically next to the edges.
Ans. I'll denote an by its weight and its vertices. The tree is unique, but the method of obtaining it is not. If we use the greedy algorithm that chooses edges of least weight as long as cycles are not created, we choose edges in the order

1:(A,G), 2:(C,D), 3:(A,B), 4:(A,D), 8:(D:F), 11:(D,E).

If we use the algorithm that adds least weight edges one by one to grow a tree, the answer depends on where we start. If we start with vertex A, we add edges as follows

1:(A,G), 3:(A,B), 4:(A,D), 2:(D,C), 8:(D,F), 11:(D,E).

On the other hand, if we start with vertex E, say, we get

11:(E,D), 2:(D,C), 5:(C,A), 1:(A,G), 3:(A,B), 8:(D,F).

(See the exam for a copy of the graph.)

2. For $k > 0$, let $p_k(n)$ be the number of partitions of the integer n into at most k nonzero parts. Define $p_k(0) = 1$ For example, $p_3(6) = 7$ because

$$6 = 1 + 5 = 2 + 4 = 3 + 3 = 1 + 1 + 4 = 1 + 2 + 3 = 2 + 2 + 2.$$

(a) Prove that $p_k(n) = \begin{cases} p_{k-1}(n), & \text{if } n < k, \\ p_{k-1}(n) + p_k(n-k), & \text{if } n \geq k. \end{cases}$

Hint: If there are k nonzero parts, decrease each part by 1.
Ans. If $n < k$, the partition must have less than k parts and so is counted by $p_{k-1}(n)$. Suppose $n \geq k$. EITHER the partition has less than k parts and so is counted by $p_{k-1}(n)$ OR the partition has *exactly* k parts. In the latter case, decrease each part by 1 to obtain a partition of $n-k$ with at most k parts. These are counted by $p_k(n-k)$. (To be correct, we need $p_k(0) = 1$ for the case in which $n = k$ is partitioned into k 1's.

(b) Using the result in (a) and the fact that $p_1(n) = 1$ for $n \geq 0$, prove by induction that

$$p_2(n) = \begin{cases} \dfrac{n+2}{2}, & \text{if } n \text{ is even,} \\ \dfrac{n+1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Hint: Do $n = 0$ and $n = 1$ separately before starting the induction.

Ans. For $n = 0$ or 1, we have $p_2(n) = p_1(n)$ by (a). Since $p_1(n) = 1$, this agrees with the formula for $p_2(n)$ in the problem. Suppose $n > 1$. By (a) and the induction hypothesis,

$$p_2(n) = 1 + p_2(n-2) = \begin{cases} 1 + \dfrac{(n-2)+2}{2} = \dfrac{n+2}{2}, & \text{if } n \text{ is even,} \\ 1 + \dfrac{(n-2)+1}{2} = \dfrac{n+1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

3. Consider the (strictly) decreasing functions from $\underline{3}$ to $\underline{11}$.

(a) Show that there are 165 of them.

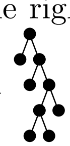
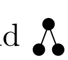
Ans. Since the decreasing functions correspond to subsets, there are $\binom{11}{3} = 165$ of them.

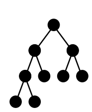
Alternatively, the rightmost function is the largest, namely 11,10,9. By the rank formula, its rank is 164 and so there are 165 functions.

(b) If these functions are arranged in lex order, find the function that is in the exact middle of the list.

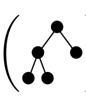
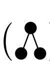
Ans. By (a), the middles is the 83rd one; that is, the one of rank 82. From Theorem 3.4, we need $82 = \binom{f(1)-1}{3} + \binom{f(2)-1}{2} + \binom{f(3)-1}{1}$, which is $\binom{8}{3} + \binom{7}{2} + \binom{5}{1}$, and so the function is 9,8,6.

4. (a) Find the 7-leaf binary RP-tree whose rank is 77.

Ans. Since $0 \leq 77 - (b_1 b_6 + b_2 b_5 + b_3 b_4 + b_4 b_3) = 1 < b_5 b_2$, the left child of the root leads to a tree with 5 leaves and the right to one with 2. Since $1 = 1 \times b_2 + 0$, the left tree has rank 1 and the right has rank 0. These trees can be found by similar calculations. We obtain  for the left tree and  for the right.

(b) Find the rank of the tree 

Ans. The rank is

$b_1 b_4 + b_2 b_3 + \text{RANK}\Big($  $\Big)\, b_2 + \text{RANK}\big($  $\big) = 1 \times 5 + 1 \times 2 + 1 \times 1 + 0 = 8.$