1. (a) How many four card hands are there where no cards are in the same suit and no cards are the same kind?
   Ans. Choose a subset of the 13 kinds and then assign the suits to them: $\binom{13}{4} \times 4!$. Alternatively, list the suits in some arbitrary order (e.g., alphabetical: clubs, diamonds, hearts, spades), then choose an ordered list of 4 kinds to go with this: $13 \times 12 \times 11 \times 10$.

   (b) How many 6 card hands contain two 3-of-a-kinds; that is, two sets of cards where each set consists of 3 cards of the same kind?
   Ans. Choose the two 3-of-a-kind values and then choose 3 suits for each: $\binom{13}{2} \times \binom{4}{3}^2 = \binom{13}{2} \times 4^2$.

2. (a) Give an example of a SIMPLE graph with a cut vertex and indicate the cut vertex.

   (b) For the graph in (a), indicate a walk which is NOT a path.

   Ans. There are many possible answers to (a) and (b).

   (c) Give an example of a SIMPLE graph with 3 vertices and 12 edges.
   Ans. The number of possible edges in a simple graph is $\binom{n}{2}$, which is 3 when $n = 3$. Hence THERE ARE NO EXAMPLES.

3. An *oriented* simple graph is a simple graph which has been converted to a digraph by assigning an orientation to each edge.

   (a) Prove that the number of n-vertex oriented simple graphs is $3^{\binom{n}{2}}$.
   Ans. Between each pair of vertices there are 3 choices (no edge, oriented one way, or oriented the other). There are $\binom{n}{2}$ pairs of vertices.

   (b) State and prove a formula for then number of n-vertex oriented simple graphs that have exactly q edges.
   Hint: You can construct an oriented simple graph by choosing a simple graph and then orienting each of its edges.
   Ans. There are $\binom{\binom{n}{2}}{q}$ simple graphs with q edges. Since each edge can be oriented in two ways, we obtain $\binom{\binom{n}{2}}{q}2^q$.

4. The *depth* of a rooted tree is the number of edges in the longest path from the root to a leaf. A *binary* rooted tree is a rooted tree in which each vertex either is a leaf or has exactly two children. Some examples are on the blackboard.

   (a) Let L_n be the maximum number of leaves in a binary rooted tree of depth n. Prove that

   $$L_0 = 1 \quad \text{and} \quad L_n = 2L_{n-1} \text{ for } n > 0.$$

   Hint: What happens when the root is removed?
   Ans. Since the only depth 0 tree is a single vertex, which is a leaf, $L_0 = 1$. When we remove the root, we obtain two trees where each has depth less than n and at least one has depth $n - 1$. To get the maximum number of leaves, we make both trees of depth $n - 1$ and with the maximum number of leaves.

(b) Using (a), prove that $L_n = 2^n$ for $n \geq 0$.
Ans. The formula is correct for $n = 0$. By the recursion and then the induction hypothesis, $L_n = 2L_{n-1} = 2(2^{n-1}) = 2^n$.

(c) Let l_n be the minimum number of leaves in a binary rooted tree of depth n. Prove that $l_n = n + 1$.
Ans. Look at the answer to (a). Now we want to choose one tree a small as possible (a single vertex) and so $l_n = l_{n-1} + 1$ for $n > 0$. Again, $l_0 = 1$ and the result follows by induction. (You could also have proved this by drawing a picture of what the minimum-leaf tree would look like.)