Math 184A First Midterm OPEN BOOK 2/3/99

Each problem is worth 12 points. Please start each problem on a new page.

1. (a) How many four card hands are there where no cards are in the same suit and no cards are the same kind? ("Kind" means A, 2, 3, ..., 10, J, Q, K.)
   (b) How many 6 card hands contain two 3-of-a-kinds; that is, two sets of cards where each set consists of 3 cards of the same kind?

2. (a) Give an example of a SIMPLE graph with a cut vertex and indicate the cut vertex.
   (b) For the graph in (a), indicate a walk which is NOT a path.
   (c) Give an example of a SIMPLE graph with 3 vertices and 12 edges.

3. An *oriented* simple graph is a simple graph which has been converted to a digraph by assigning an orientation to each edge.
   (a) Prove that the number of n-vertex oriented simple graphs is $3^{\binom{n}{2}}$.
   (b) State and prove a formula for the number of n-vertex oriented simple graphs that have exactly q edges.
   Hint: You can construct an oriented simple graph by choosing a simple graph and then orienting each of its edges.

4. The *depth* of a rooted tree is the number of edges in the longest path from the root to a leaf. A *binary* rooted tree is a rooted tree in which each vertex either is a leaf or has exactly two children. Some examples are on the blackboard.
   (a) Let L_n be the maximum number of leaves in a binary rooted tree of depth n. Prove that
   $$L_0 = 1 \quad \text{and} \quad L_n = 2L_{n-1} \text{ for } n > 0.$$
   Hint: What happens when the root is removed?
   (b) Using (a), prove that $L_n = 2^n$ for $n \geq 0$.
   (c) Let l_n be the minimum number of leaves in a binary rooted tree of depth n. Prove that $l_n = n + 1$.