1. We can either use no letters twice, A twice, T twice, or both letters twice. In any case, we must choose the placement of the repeated letters, choose the single letters, and choose the order for placing the single letters. Here's what we get

$$\begin{aligned} \text{no repeats:}\quad & A_0 = \binom{7}{n} n! = \frac{7!}{(7-n)!} \\ \text{A repeats:}\quad & A_1 = \binom{n}{2}\binom{6}{n-2}(n-2)! = \frac{(n)_2\, 6!}{2\,(8-n)!} \qquad \text{[same for T]} \\ \text{A \& T repeat:}\quad & A_2 = \binom{n}{2}\binom{n-2}{2}\binom{5}{n-4}(n-4)! = \frac{(n)_4\, 5!}{4\,(9-n)!}. \end{aligned}$$

You could also do this with multinomial coefficients, but I haven't talked about them much. The final answer is $A_0 + 2A_1 + A_2$, where the "2" comes from the choice of A or T. One could combine the terms, but need not—it gives a mess.

2. I'll do $P_k(n+1)$. If it is in a cycle of length $j+1$, we can complete the cycle by listing, in order, the j elements that follow it. We then form a permutation on the remaining $n-j$ elements. Thus we have $n(n-1)\cdots(n-j+1)P_k(n-j) = (n)_j P_k(n-j)$. This is fine if $n > j$, but what about the other cases? If $j > n$, there are not enough elements so the value should be zero and it is since $P_k(n-j) = 0$ when $n-j < 0$. If $j = n$, there are no remaining elements to permute so we have just the $(j+1)$-cycle and so should multiply by 1. Again, this is okay since $P_k(0) = 1$. Adding up we get

$$P_k(n+1) = \sum_{j=0}^{k-1} (n)_j P_k(n-j).$$

3. (a) The definition is (a vertex) OR (a vertex AND a left tree) OR (a vertex and a right tree) OR (a vertex AND a left tree AND a right tree). Clearly no tree is obtained more than once, so by the Rules of Sum and Product we have $S(x) = x + xS(x) + xS(x) + xS(x)S(x)$.

   (b) Writing S for $S(x)$, we have $S = x + 2xS + xS^2$ and so $xS^2 - (1-2x)S + x = 0$. The solution is

   $$S(x) = \frac{1 - 2x \pm \sqrt{1-4x}}{2x}.$$

   Since there is no constant term, the correct sign is minus.

   (c) From memory or by solving the given equation,

   $$B(x) = \frac{1 - \sqrt{1-4x}}{2} = xS(x) - x.$$

   Equating coefficients of x^{n+1} for $n > 0$ gives $b_{n+1} = s_n$.

4. (a) This is just a restatement of the definition using the Rules of Sum and Product, once we note that each sequence is obtained only once in the definition.

   (b) Since z_0 counts only the empty sequence, it is 1. For $n > 0$ a zero-free sequence is formed by first choosing any of $1, \ldots, k$ and thereafter choosing a number different from the one just chosen. This gives

$$z_n = k(k-1)\cdots(k-1) = k(k-1)^{n-1}.$$

   (c) From (a), $S(x) = Z(x)/(1 - xZ(x))$, so we need a formula for $Z(x)$. We have

$$\begin{aligned} Z(x) &= 1 + \sum_{n=1}^{\infty} k(k-1)^{n-1}x^n = 1 + kx\sum_{n=1}^{\infty}((k-1)x)^{n-1} \\ &= 1 + \frac{kx}{1-(k-1)x} = \frac{1+x}{1-(k-1)x}. \end{aligned}$$

   Hence

$$S(x) = \frac{\frac{1+x}{1-(k-1)x}}{1 - \frac{x(1+x)}{1-(k-1)x}} = \frac{1+x}{(1-(k-1)x) - x(x+1)} = \frac{1+x}{1-kx-x^2}.$$

5. (a) Use induction on n. For $n = 0$, the last fact tells us that the graph is a tree and has no cycles. Suppose $n > 0$. Since the last fact is false, the second fact is too; that is, the graph has cycles. Let e be an edge on a cycle. Removing e does not disconnect the graph but destroys any cycle containing it, so the new graph has at least one less cycle than the original graph. The removal gives a simple connected graph with v vertices and $v+n-1$ edges. By the induction hypothesis, this graph has at least $(n-1)+1 = n$ cycles and so the original graph had at least n.

   The fact that removing an edge in a cycle does not disconnect a graph was proven in class. If you want to see a proof, here's one. Let $e = \{x, y\}$. Using the cycle, there is a path P from x to y that does not use e. Any walk in the original graph that uses e can be converted into a new walk that does not by replacing e with either P or its reverse.

   (b) Looking closely at the proof, we can see that we get the minimum number of cycles if removing an edge on a cycle destroys only one cycle. That can be used to construct a graph. On the other hand, one could also use the hint, which suggests stringing $n+1$ triangles together in some fashion. Any way of attaching them is fine as long as you don't construct a "cycle" of triangles.