1. The rank formula is $\sum_{i=1}^{k}\binom{f(i)-1}{k-i+1}$ with $k=4$. In other words,

$$\binom{f(1)-1}{4}+\binom{f(2)-1}{3}+\binom{f(3)-1}{2}+\binom{f(4)-1}{1}.$$

(a) Since $\binom{7}{4}=35\le 50<\binom{8}{4}$, $f(1)=8$ and we need a function on $\underline{3}$ of rank 15. Since $\binom{5}{3}=10\le 15<\binom{6}{3}$, $f(2)=6$ and we need a function on $\underline{2}$ of rank 5. Since $\binom{3}{2}=3\le 5<\binom{4}{2}$, $f(3)=4$ and we need a function on $\underline{1}$ of rank 2. Thus $f(4)=3$ and the function is 8,6,4,3.

(b) Using the formula for 7,4,2,1 gives $\binom{6}{4}+\binom{3}{3}+\binom{1}{2}+\binom{0}{1}=16$.

2. We have $20=b_1b_5+b_2b_4+b_3\times 0+1$. Thus the left tree has 3 vertices and rank 0 while the right has 3 vertices and rank 1. Since there are only two 3-vertex binary trees, we easily obtain the answer.

3. (a) They are obtained by starting with a triangle on the vertices 1,2,3 and then joining vertex 4 to exactly two of 1,2,3. In other words add one of the following three sets to E:

$$\Big\{\{1,4\},\ \{2,4\}\Big\}\qquad\Big\{\{1,4\},\ \{3,4\}\Big\}\qquad\Big\{\{2,4\},\ \{3,4\}\Big\}$$

(b) Every web is built from the triangle, which contains a cycle. Since a tree does not contain a cycle, a web cannot be a tree.
Alternate proof: Every tree (except the single point) contains vertices of degree 1, but every vertex in a web has degree at least 2.

(c) We have $w_3=1$ by (i) and, for $n>3$, $w_n=\binom{n-1}{2}w_{n-1}$ because we can choose a web on $n-1$ vertices AND choose two vertices to connect to vertex n in $\binom{n-1}{2}$ ways.

(d) We use induction. The case $n=3$ is simple to check. For $n>3$,

$$\begin{aligned} w_n &= \binom{n-1}{2}w_{n-1} && \text{by (c)}\\ &= \frac{(n-1)(n-2)}{2}\,\frac{(n-1-1)!\,(n-2-2)!}{2^{n-1-2}} && \text{by induction}\\ &= \frac{(n-1)!\,(n-2)!}{2^{n-2}}. \end{aligned}$$