- Please put your name and ID number on your blue book.
- CLOSED BOOK except for BOTH SIDES of one page of notes.
- Calculators are NOT allowed.
- **You must show your work to receive credit.**

1. (18 pts.) Let $S(n,k)$ be the Stirling numbers of the second kind — the number of partitions of an n-set into k blocks. Derive the following

   (a) $S(n,2) = 2^{n-1} - 1$ (b) $S(n,n-1) = \dfrac{n(n-1)}{2}$.

2. (10 pts.) A long rectangular table has n seats on each side and none at the ends. We want to seat n men and n women at the table so that no one is next to or opposite a person of the same sex. How many ways can this be done?
   Note: There are no symmetries—the sides of the table are different.

3. (14 pts.) Each edge of a simple graph G is assigned a weight from the set $\{1, 2, \ldots, 9\}$ and each of the nine weights is used at least once.

   (a) If the graph G has 10 (ten) edges, what is the largest number of minimum weight spanning trees it can have? Why?

   (b) If the graph G has 11 (eleven) edges, what is the largest number of minimum weight spanning trees it can have? Why?

   Hint: Recall that two minimum weight spanning trees have the same multiset of edge weights.

4. (10 pts.) The chromatic polynomial of a graph G is $P_G(x)$. It is known that $P_G(x) = 0$ for $x = 0$, 1, 2 and 3. How many ways can G be (properly) colored using 6 (six) colors if every color *must* be used? Express your answer in terms of $P_G(4)$, $P_G(5)$ and $P_G(6)$.

5. (18 pts.) Here is a local description of the solution to the Tower of Hanoi puzzle.

   $$\texttt{H}(1,S,E,G) = S \xrightarrow{1} G$$

   $n > 1$: $\texttt{H}(n,S,E,G)$

   $$\texttt{H}(n-1,S,G,E) \quad S \xrightarrow{n} G \quad \texttt{H}(n-1,E,S,G)$$

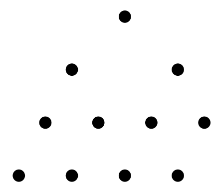
   Because of washer weight, the work to move washer k is k^2.

   (a) Write down a recursion for the amount of work to solve the puzzle, including the initial condition(s).

   (b) Show that the amount of work required for n washers is $6 \times 2^n - n^2 - 4n - 6$.

THERE ARE MORE PROBLEMS

6. (10 pts.) With the standard ordering, find the rank of the 6-leaf full binary tree shown here.



Here are the first few values of b_n, the number of such trees with n leaves,

$$b_1 = b_2 = 1 \quad b_3 = 2 \quad b_4 = 5 \quad b_5 = 14 \quad b_6 = 42.$$

7. (10 pts.) Consider unlabeled RP-trees where each non-leaf vertex must have an odd number of children. Let t_n be the number with n vertices and let $T(x) = \sum_{n=1}^{\infty} t_n x^n$. Derive the formula $T(x)^3 - xT(x)^2 + (x-1)T(x) + x = 0$.

8. (10 pts.) A certain exponential generating function $G(x) = \sum_{n=1}^{\infty} g_n x^n/n!$ satisfies the equation $G(x) = xe^{G(x)}$. Find A, b and r so that $g_n \sim (An^b r^{-n})\, n!$.

Principle 11.6 (Nice singularities, shortened) Let a_n be a sequence whose terms are positive for all suffciently large n. Suppose that $A(x) = \sum_n a_n x^n$ converges for some value of $x > 0$. Suppose that $A(x) = f(x)g(x) + h(x)$ where

- $f(x) = (1 - x/r)^c$, c is not a positive integer or zero;
- $g(r) \neq 0$ and $g(x)$ does not have a singularity at $x = r$;
- $A(x)$ does not have a singularity for $-r \le x < r$;
- $h(x)$ does not have a singularity at $x = r$.

Then it is usually true that

$$a_n \sim \frac{g(r)(1/r)^n}{n^{c+1}\Gamma(-c)}$$

where

$$\Gamma(k) = (k-1)! \text{ when } k > 0 \text{ is an integer,} \quad \Gamma(x+1) = x\Gamma(x) \quad \text{and} \quad \Gamma(1/2) = \sqrt{\pi}.$$

Principle 11.7 (Implicit functions) Let a_n be a sequence whose terms are positive for all suffciently large n. Let $A(x)$ be the ordinary generating function for the a_n's. Suppose that the function $F(x, y)$ is such that $F(x, A(x)) = 0$. If there are positive real numbers r and s such that $F(r, s) = 0$ and $F_y(r, s) = 0$ and if r is the smallest such r, then it is usually true that

$$a_n \sim \sqrt{\frac{rF_x(r,s)}{2\pi F_{yy}(r,s)}}\, n^{-3/2} r^{-n}.$$

END OF EXAM