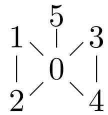
Math 184A Second Exam Solutions 2 March 2005

1.  You can use theorems, but it is probably easiest to color 0 (x ways), then color 1, 3 and 5 ($x-1$ ways each since each must differ from 0) and finally color 2 and 4 ($x-2$ ways each since each must differ from two differently colored vertices—0 and 1 or 0 and 3). This gives $x(x-1)^3(x-2)^2$.

2. (a) Use induction. True for $n=1$ and $n=2$ by inspection. For $n>2$, the left (resp. right) tree has leaves of length $1+(n-1)$ (resp. $2+(n-2)$).

   (b) It is evident that no leaf has adjacent B's since we either prepend A or BA. To see that every such sequence arises, use induction. It's true for lengths 1 and 2 by inspection. For $n>2$, the sequence must begin either A or BA and then is followed by any sequence with no adjacent B's. By the induction hypothesis for $n-1$ and n all such sequences occur in $S^*(n-1)$ and $s^*(n-2)$.

3. (a) We have $(1-x-2x^2)A(x)=x$. Equating coefficients of x^n, we have

$$a_n - a_{n-1} - 2a_{n-2} \text{ equals 3 if } n=1 \text{ and 0 otherwise.}$$

   Treating $n \le 1$ as initial conditions, this gives us

$$a_0 = 0, \quad a_1 = 3, \quad a_n = a_{n-1} + 2a_{n-2} \ \text{ for } n \ge 2.$$

   Alternatively, introducing $c_n = 0$ except that $c_1 = 3$ gives us

$$a_n = a_{n-1} + 2a_{n-2} + c_n \ \text{ for all } n.$$

   (As usual, a_n is assumed zero for negative n.)

   (b) Since $1-x-2x^2 = (1-2x)(1+x)$, partial fractions gives us

$$A(x) = \frac{b}{1-2x} + \frac{c}{1+x} = \sum b(2x)^n + \sum c(-x)^n = \sum (b2^n + c(-1)^n)x^n.$$

   Thus $a_n = b2^n + c(-1)^n$. You can find $b=1$ and $c=-1$ either by the usual partial fraction route or by solving the two equations $0 = a_0 = b+c$ and $3 = 2b-c$.

4. Each such tree is either a single vertex (xy) OR a root (x) joined to two trees ($T(x,y)^2$) OR a root joined to four trees, etc. Thus we have

$$T(x,y) = xy + \sum_{\substack{k\ge1 \\ k \text{ even}}} x(T(x,y))^k = xy + \frac{x(T(x,y))^2}{1-(T(x,y))^2}.$$