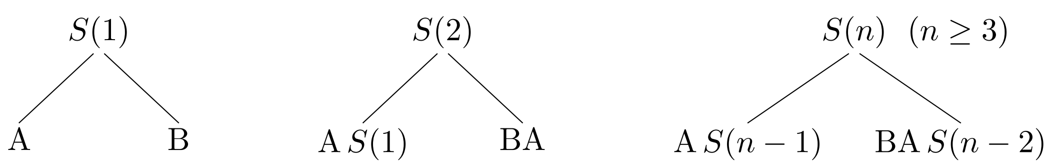
Math 184A | Second Exam (50 points) | 2 March 2005

- Please put your name and ID number on your blue book.
- The exam is CLOSED BOOK except one 2-sided page of notes.
- Calculators are NOT allowed.
- **You must show your work to receive credit.**

1. (10 pts.) Let G be the simple graph with vertex set $\{0,1,2,3,4,5\}$ and edges $\big\{\{0,1\},\ \{0,2\},\ \{0,3\},\ \{0,4\},\ \{0,5\},\ \{1,2\},\ \{3,4\}\big\}$.
   Sketch G and compute $P_G(x)$, the chromatic polynomial of G.

2. (12 pts.) The local description of a decision tree for constructing sequences of A's and B's is given below. The notation BA $S(n-2)$ means place BA in front of each sequence produced by $S(n-2)$.

   Let $S^*(n)$ denote the entire decision tree. Thus $S^*(1) = S(1)$ and $S^*(2)$ has the three leaves AA, AB, and BA.

   (a) Prove that each leaf of $S^*(n)$ is an n-long sequence of A's and B's.

   (b) Prove that each non-empty sequence of A's and B's will be the leaf of some $S^*(n)$ if and only if the sequence does not contain two B's in a row. For example, ABABAAB is a leaf but ABBABA is not a leaf.

3. (18 pts.) It is known that the ordinary generating function for the sequence $a_0, a_1, a_2, \ldots$ is given by
   $$A(x) = \frac{3x}{1 - x - 2x^2}.$$

   (a) Derive a simple recursion for the sequence. Be sure to include initial conditions.

   (b) Derive a simple formula for a_n.

4. (10 pts.) Consider unlabeled, rooted plane trees in which each vertex has an even number of downward edges (and, of course, one upward edge if it is not the root). Let $t_{n,k}$ be the number of such trees with n vertices and k leaves. Let $T(x,y)$ be the generating function $\sum t_{n,k} x^n y^k$.

   $$\text{Derive the formula} \quad T(x,y) \;=\; xy + \frac{x(T(x,y))^2}{1 - (T(x,y))^2}.$$

   Example: Full binary trees are examples of such trees since each vertex has either two or zero downward edges.

END OF EXAM