1. (a) We'll do this in three different ways.
   (i) Count the number of rearrangements of $1, 2, \ldots, 8$ in two ways: First there are $8!$. Second, seat people at a table, pick a side, and read off a list starting at that side and going clockwise around the table. Thus $8! = (\text{answer}) \times 4$, giving $8!/4$.
   (ii) Designate a first person. Place that person on one side of the table (2 possible seats) and then arrange the remaining 7 people, giving $2 \times 7!$.
   (iii) Use Burnside's Lemma. The rotations are $0°$, $90°$, $180°$ and $270°$. $N(0°) = 8!$ and, since all the people are different, $N(r°) = 0$ for $r \neq 0$. Thus we have $\frac{1}{4}(8! + 0 + 0 + 0 + 0)$.

   (b) This could be done in various ways. The easiest is to use Burnside's Lemma. Call positions around the table $1, 2, \ldots, 8$ reading clockwise as shown in the first picture in the exam problem. The group elements in cycle form are

   no rotation: $(1)(2)(3)(4)(5)(6)(7)(8)$ 90° rotation: $(1,3,5,7)(2,4,6,8)$
   180° rotation: $(1,5)(3,7)(2,6)(4,8)$ 270° rotation: $(1,7,5,3)(2,8,6,4)$

   Since chairs must be the same (color) on a cycle, we choose which cycles should have red chairs, getting the answer

$$\frac{1}{4}\left[\binom{8}{4} + \binom{2}{1} + \binom{4}{2} + \binom{2}{1}\right] = \frac{70+2+6+2}{4} = 20.$$

   The other way is to attempt to list all 20 solutions, but it is very easy to omit a solution or count it twice.

2. (a) There are $N = \binom{n}{2}$ possible edges. Since we must choose q of them, the answer is $\binom{N}{q}$.
   (b) Since the vertices in S cannot be used, there are $M = \binom{n-|S|}{2}$ possible edges and the answer is $\binom{M}{q}$.
   (c) Use (b) and the Principle of Inclusion and Exclusion.

3. There is no such graph. Suppose G were such a graph. We can construct a spanning tree T for G by removing edges one at a time. Using subscripts to indicate the number of edges, we obtain the sequence $G = G_{25}, G_{24}, G_{23}, G_{22}, G_{21}, G_{20}, G_{19} = T$ since a tree has one less edge than it has vertices. Thus we removed six edges. When an edge e_i is removed from G_i, it must belong to at least one cycle of G_i (since otherwise G_{i-1} would not be connected). Thus, removing e_i destroys at least one cycle of G_i and hence of G. Since we removed six edges, we must have destroyed at least six cycles of G. Thus G must have at least six cycles.

4. Apply Principle 11.7 with $F(x,y) = x(e^y - y) - y$. Thus we must solve the pair of equations

$$F(r,s) = r(e^s - s) - s = 0 \qquad \text{and} \qquad F_y(r,s) = r(e^s - 1) - 1 = 0.$$

Multiply the second by s and subtract the first to obtain $rse^s - re^s = 0$. Thus $s = 1$ and, by either the first or second displayed equation, $r = (e-1)^{-1}$. Hence we have $t_n/n! \sim An^{-3/2}(e-1)^n$.

5. (a) Let the amount of work be w_n. From the local description $w_1 = 1$ and $w_n = 2w_{n-1} + n$ when $n > 1$. With the condition that $w_n = 0$ for $n \leq 0$, we can combine these into one recursion: $w_n = 2w_{n-1} + n$ when $n \geq 0$.

   (b) One way to do this is to obtain the generating function and expand it. Here is the result, without details:

$$W(x) = \frac{x}{(1-x)^2(1-2x)} = \frac{2}{1-2x} - \frac{1}{(1-x)^2} - \frac{2}{1-x}$$

   and so $w_n = 2^{n+1} - n - 2$.

   Another approach is to prove the formula by induction using the recursion. It is easily checked when $n = 1$. For $n > 1$,

$$2w_{n-1} + n = 2(2^n - (n-1) - 2) = 2^{n+1} - 2n - 2.$$

6. A tree is either a single vertex OR two trees joined to a root OR three trees joined to a root, except that we cannot have all tree be non-leaves. Thus we have

$$T(x) = x + T(x)^2 + T(x)^3 - S(x),$$

   where $S(x)$ is the situation in which three trees, none of which are leaves, are joined to a root. Since the generating function for trees which are not leaves is $T(x) - x$, we have $S(x) = (T(x) - x)^3$.