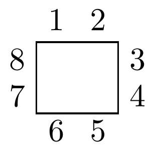
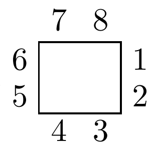
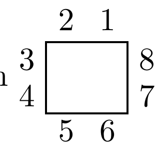
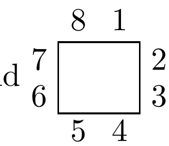
- Please put your name and ID number on your blue book.
- CLOSED BOOK except for BOTH SIDES of one page of notes.
- Calculators are NOT allowed.
- **You must show your work to receive credit.**

1. (16 pts.) A square table has two seats on each side for a total of eight seats. Rotations of the table don't matter. Thus, if $1, 2, \ldots, 8$ are placed around the table,



and


are the same, but differ from


and


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   (a) How many ways can eight people be seated at the table?

   (b) We have four identical red chairs and four identical blue chairs. How many ways can the eight chairs be placed around the table? Again, rotations of the table do not matter.

2. (18 pts.) Let $V = \{1, 2, \ldots, n\}$.

   (a) Compute the number of simple graphs with vertex set V that have exactly q edges.

   (b) A vertex is *isolated* if it does not lie on any edges. Suppose $S \subset V$. Compute the number of simple graphs with vertex set V that have exactly q edges such that all the vertices in S are isolated.

   (c) Prove that the number of simple graphs with vertex set V that have exactly q edges and no isolated vertices is given by

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{\binom{n-k}{2}}{q}.$$

3. (10 pts.) I claim it is possible to construct a *connected simple graph* that has 20 vertices 25 edges and at most 5 cycles. Gregg claims that this is impossible.

   If I am right, construct such a graph. If Gregg is right, prove that there is no such graph.

   Note: Different cycles may have edges in common. For example, the simple graph with $V = \{1, 2, 3, 4\}$ and $E = \big\{\{1,2\},\ \{1,3\},\ \{4,2\},\ \{4,3\},\ \{2,3\}\big\}$ has three cycles. The vertices on the cycles are 1,2,3 and 4,2,3 and 1,2,4,3.

4. (10 pts.) The EGF for certain trees satisfies the equation

$$\sum_{n=1}^{\infty} \frac{t_n\, x^n}{n!} \;=\; T(x) \;=\; x\big(e^{T(x)} - T(x)\big).$$

   It is known that $t_n \sim A\, n^b\, C^n\, n!$. Find b and C.

5. (16 pts.) Here is a local description of the solution to the Tower of Hanoi puzzle.

$$\texttt{H}(1,S,E,G) \;=\; S \xrightarrow{1} G$$

$n > 1$: $\texttt{H}(n,S,E,G)$ has three children:

$$\texttt{H}(n-1,S,G,E) \quad S \xrightarrow{n} G \quad \texttt{H}(n-1,E,S,G)$$

Because of washer weight, the work to move washer k is k.

(a) Write down a recursion for the amount of work to solve the puzzle.

(b) Show that the amount of work required for n washers is $2^{n+1} - n - 2$.

6. (10 pts.) Consider unlabeled RP-trees in which each non-leaf vertex can have two or three children and, if it has three, one of the children must be a leaf. Let t_n be the number with n leaves. Derive the equation

$$\sum_{n=1}^{\infty} t_n x^n \;=\; T(x) \;=\; x + T(x)^2 + T(x)^3 - \big(T(x) - x\big)^3.$$

Principle 11.6 (Nice singularities, shortened) Let a_n be a sequence whose terms are positive for all suffciently large n. Suppose that $A(x) = \sum_n a_n x^n$ converges for some value of $x > 0$. Suppose that $A(x) = f(x)g(x) + h(x)$ where

- $f(x) = (1 - x/r)^c$, c is not a positive integer or zero;
- $g(r) \neq 0$ and $g(x)$ does not have a singularity at $x = r$;
- $A(x)$ does not have a singularity for $-r \le x < r$;
- $h(x)$ does not have a singularity at $x = r$.

Then it is usually true that

$$a_n \;\sim\; \frac{g(r)(1/r)^n}{n^{c+1}\Gamma(-c)}$$

where

$$\Gamma(k) = (k-1)! \text{ when } k > 0 \text{ is an integer}, \quad \Gamma(x+1) = x\Gamma(x) \quad \text{and} \quad \Gamma(1/2) = \sqrt{\pi}.$$

Principle 11.7 (Implicit functions) Let a_n be a sequence whose terms are positive for all suffciently large n. Let $A(x)$ be the ordinary generating function for the a_n's. Suppose that the function $F(x, y)$ is such that $F(x, A(x)) = 0$. If there are positive real numbers r and s such that $F(r, s) = 0$ and $F_y(r, s) = 0$ and if r is the smallest such r, then it is usually true that

$$a_n \;\sim\; \sqrt{\frac{rF_x(r,s)}{2\pi F_{yy}(r,s)}}\, n^{-3/2} r^{-n}.$$

END OF EXAM