Math 184A Second Exam (50 points) 21 November 2005

- Please put your name and ID number on your blue book.
- The exam is CLOSED BOOK except for one page of notes.
- Calculators are NOT allowed.
- **You must show your work to receive credit.**

1. (15 pts.) Answer each of the following TRUE or FALSE. In each case, $G = (V, E)$ is a simple graph with n vertices. *You do NOT need to give reasons for your answers.*

   (a) If G has $n - 1$ edges, it must be a tree.

   (b) If the edges of G have weights (also known as costs) and there are two minimum weight spanning trees, then two edges have the same weight.

   (c) If $u, v \in V$ and there are two paths connecting u and v, then G contains a cycle. (The two paths may share edges and/or vertices, but they are not two copies of the same path.)

   (d) If $u, v \in V$ and there are two paths connecting u and v and they have no edges in common, then there is a cycle containing both u and v.

   (e) Let $P_G(x)$ be the chromatic polynomial of G. If $P_G(10) = 0$, then $P_G(3) = 0$.

2. (5 pts.) Let G be the graph with vertices $\{0, 1, 2, 3, 4, 5, 6\}$. Its 9 edges are

$$\{0,1\},\ \{0,2\}, \ldots,\ \{0,6\} \quad \text{and} \quad \{1,2\},\ \{3,4\},\ \{5,6\}.$$

   Sketch G and compute its chromatic polynomial.

3. (10 pts.) Suppose that G is a simple graph with 25 vertices.

   (a) Prove: If G has 24 edges and is not connected, then G has a cycle.

   (b) Show that the result in (a) would be false if "24" were replaced by "23".

4. (10 pts.) Let $s_k(r) = \sum_{n=0}^{\infty} \binom{2n}{k} r^{2n}$.

   (a) Derive the formula

$$\sum_{k=0}^{\infty} s_k(r) z^k \;=\; \frac{1}{2}\left(\frac{1}{1-(1+z)r} + \frac{1}{1+(1+z)r}\right).$$

   Hint: Recall that $\sum_{n,k} \binom{n}{k} x^n y^k = \frac{1}{1-(1+y)x}$.

   (b) Obtain a simple formula for $s_k(1/2)$.

5. (10 pts.) Let t_n be the number of n-leaf unlabeled, rooted plane trees such that no vertex has exactly one child.

   (a) Find an equation satisfied by $T(x) = \sum_{n=1}^{\infty} t_n\, x^n$.

   (b) Find the generating function for $T(x)$.
   (If you have the correct answer, it will contain a square root.)

END OF EXAM