1. (a) Any permutation containing a 3-cycle will work.
   (b) Since an n-vertex simple graph has at most $\binom{n}{2}$ edges, it is impossible.

2. (a) The split must be 3–4, so we have

$$\binom{5}{3}\binom{8}{4}+\binom{5}{4}\binom{8}{3} \;=\; 980.$$

   (b) The splits must be 3–4 and 2–3. Since there are only 5 Democrats, they all serve on a committee and both committees must have a majority of Republicans. Choose the first committee and then choose the Republicans for the second:

$$\binom{5}{3}\binom{8}{4}\binom{4}{3} \;=\; 2800.$$

3. (a) We'll do this in three different ways.
   (i) Count the number of rearrangements of $1, 2, \ldots, 8$ in two ways: First there are $8!$. Second, seat people at a table, pick a side, and read off a list starting at that side and going clockwise around the table. Thus $8! = (\text{answer}) \times 4$, giving $8!/4$.
   (ii) Designate a first person. Place that person on one side of the table (2 possible seats) and then arrange the remaining 7 people, giving $2 \times 7!$.
   (iii) Use Burnside's Lemma. The rotations are $0°$, $90°$, $180°$ and $270°$. $N(0°) = 8!$ and, since all the people are different, $N(r°) = 0$ for $r \neq 0$. Thus we have $\frac{1}{4}(8! + 0 + 0 + 0 + 0)$.
   (b) This could be done in various ways. The easiest is to use Burnside's Lemma. Call positions around the table $1, 2, \ldots, 8$ reading clockwise as shown in the first picture in the exam problem. The group elements in cycle form are

no rotation:	$(1)(2)(3)(4)(5)(6)(7)(8)$	$90°$ rotation:	$(1,3,5,7)(2,4,6,8)$
$180°$ rotation:	$(1,5)(3,7)(2,6)(4,8)$	$270°$ rotation:	$(1,7,5,3)(2,8,6,4)$

   Since chairs must be the same (color) on a cycle, we choose which cycles should have red chairs, getting the answer

$$\frac{1}{4}\left[\binom{8}{4}+\binom{2}{1}+\binom{4}{2}+\binom{2}{1}\right] \;=\; \frac{70+2+6+2}{4} \;=\; 20.$$

   The other way is to attempt to list all 20 solutions, but it is very easy to omit a solution or count it twice. If you do that, you will lose as many points as your list differs from 20, except that your score will not be negative.

4. Since $b_1b_6 + b_2b_5 = 56 \leq 60 < b_1b_6 + b_2b_5 + b_3b_4$ and $60 - 56 = 4 = 0b_4 + 4$, the left tree has 3 leaves and rank 0 and the right has 4 leaves and rank 4.
   Since the 3-leaf tree has least possible rank, it branches rightward as much as possible. In bracket notation, $[\bullet, [\bullet, \bullet]]$. (The bullets are leaves.)

Since the 4-leaf tree has greatest possible rank ($b_4 - 1$), it branches leftward as much as possible. In bracket notation, $[[[\bullet, \bullet], \bullet], \bullet]$.

5. This can be done with the Principle of Inclusion and Exclusion. Let property i be that color i was not used. For any set S of colors, the tree can be properly colored without those colors in $P_T(4 - |S|)$ ways. Thus we have

$$P_T(4) - \binom{4}{1} P_T(3) + \binom{4}{2} P_T(2) - \binom{4}{3} P_T(1) + \binom{4}{4} P_T(0) \;=\; 4\times 3^4 - 4\times 3\times 2^4 + 6\times 2\times 1^4,$$

which equals 144.

Alternatively, you can try to count them directly. Since there are five vertices and four colors, one color appears twice and the others once.
(i) Choose the repeated color (4 ways) AND
(ii) choose where to place it (6 ways, see below) AND
(iii) place the remaining 3 colors (3! ways),
giving $4\times 6\times 3!$. For (ii), there are $\binom{5}{2}$ ways to place the color if we have no constraints. There are 4 ways to place it so that the two colors are adjacent—just place the two copies at opposite ends of one of the 4 edges in the tree. Thus we get $\binom{5}{2} - 4 = 6$. The tricky part is seeing how to do (ii) for *all* 5-vertex trees. If you only did it for one tree such as a path, you can expect to lose some points.

6. C is the reciprocal of the radius of convergence. Since e^x is well behaved, the only problem is when $2 - e^x = 0$, Thus the radius of convergence is $\ln 2$ and so $C = 1/\ln 2 = \log_2 e$.
Alternatively, you could use Principle 11.6.

7. (a) Such a tree is either a single vertex (leaf), two trees joined to a new root or three threes joined to a new root. By the Rules of Sum and Product for generating functions, $T = x + T^2 + T^3$, which is equivalent to the given formula.
(b) Apply Principle 11.7 with $F(x, y) = y^3 + y^2 - y + x$. Then

$$F_y(x, y) \;=\; 3y^2 + 2y - 1.$$

From $F_y(r, s) = 0$, we have $3s^2 + 2s - 1 = 0$ and so $s = -1$ or $1/3$. Since we want the positive value, $s = 1/3$. From $F(r, s) = 0$, we have

$$(1/3)^3 + (1/3)^2 - (1/3) + x \;=\; 0$$

and so $x = 5/27$. Thus the answer is $B = -3/2$ and $C = 27/5$.