- Please put your name and ID number on your blue book.
- CLOSED BOOK, but you may have TWO PAGES of notes.
- Calculators are NOT allowed.
- You need not simplify the arithmetic in your answers.
- **You must show your work to receive credit.**

1. (8 pts.) In each case, **give an example or explain why none exists**.

   (a) A permutation f of $\{1, 2, 3, 4, 5\}$ such that f^{20} is not the indentity permutation. (The identity permutation is the function g such that $g(x) = x$ for all x in the domain.)

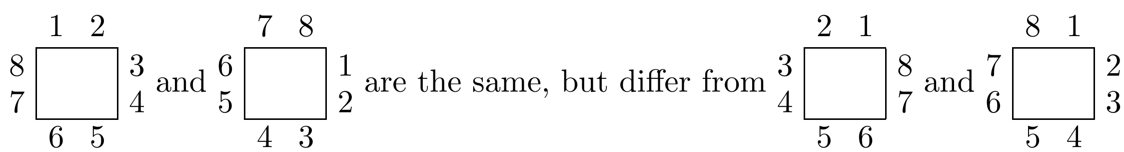
   (b) A simple graph with 5 vertices and 12 edges.

2. (16 pts.) Committees are to be formed from 5 Democrats and 8 Republicans. A committee must be as balanced as possible; that is, the number of Republicans and Democrats on the committee must be as equal as possible.

   (a) How many ways can a 7 member committee be formed?

   (b) How many ways can both a 7 member committee and a 5 member committee be formed if no person can be on both committees?

3. (16 pts.) A square table has two seats on each side for a total of eight seats. Rotations of the table don't matter. Thus, if $1, 2, \ldots, 8$ are placed around the table,



   (a) How many ways can eight people be seated at the table?

   (b) We have four identical red chairs and four identical blue chairs. How many ways can the eight chairs be placed around the table?

4. (8 pts.) Find the 7-leaf complete binary RP-tree of rank 60. Here are the numbers of trees with various leaves:

$$b_1 = b_2 = 1, \qquad b_3 = 2, \qquad b_4 = 5, \qquad b_5 = 14, \qquad b_6 = 42, \qquad b_7 = 132.$$

5. (8 pts.) The chromatic polynomial of an n-vertex tree T is $P_T(x) = x(x-1)^{n-1}$. How many ways can a 5-vertex tree be properly colored using 4 colors if *every color must be used*? Your answer should be a number, but you don't need to simplify the arithmetic.

THERE ARE MORE PROBLEMS

6. (8 pts.) Let a_n be the number of partitions of an n-set in which the blocks are ordered. It is known that

$$\sum_{n=1}^{\infty} a_n \frac{x^n}{n!} \;=\; \frac{1}{2-e^x}.$$

It can be shown that $a_n/n! \;\sim\; A\,C^n$ for some constants A and C. Find C.
Note: You are not asked to prove any of the preceding — just find C and explain how you got it.

7. (16 pts.) Consider unlabeled RP-trees where each non-leaf vertex must have either two or three children. Let t_n be the number of such trees with n leaves (with $t_0 = 0$) and let $T(x) = \sum t_n x^n$.

   (a) Derive the formula $T(x)^3 + T(x)^2 - T(x) + x \;=\; 0$.

   (b) It turns out that $t_n \;\sim\; A\,n^B\,C^n$ for some constants A, B and C. Find B and C. (Your answers should be actual numbers, not descriptions of how to find them.)

Principle 11.6 (Nice singularities, shortened) Let a_n be a sequence whose terms are positive for all suffciently large n. Suppose that $A(x) = \sum_n a_n x^n$ converges for some value of $x > 0$. Suppose that $A(x) = f(x)g(x) + h(x)$ where

- $f(x) = (1 - x/r)^c$, c is not a positive integer or zero;
- $g(r) \neq 0$ and $g(x)$ does not have a singularity at $x = r$;
- $A(x)$ does not have a singularity for $-r \leq x < r$;
- $h(x)$ does not have a singularity at $x = r$.

Then it is usually true that

$$a_n \;\sim\; \frac{g(r)(1/r)^n}{n^{c+1}\Gamma(-c)}$$

where

$$\Gamma(k) = (k-1)! \text{ when } k > 0 \text{ is an integer}, \quad \Gamma(x+1) = x\Gamma(x) \quad \text{and} \quad \Gamma(1/2) = \sqrt{\pi}.$$

Principle 11.7 (Implicit functions) Let a_n be a sequence whose terms are positive for all suffciently large n. Let $A(x)$ be the ordinary generating function for the a_n's. Suppose that the function $F(x,y)$ is such that $F(x, A(x)) = 0$. If there are positive real numbers r and s such that $F(r,s) = 0$ and $F_y(r,s) = 0$ and if r is the smallest such r, then it is usually true that

$$a_n \;\sim\; \sqrt{\frac{rF_x(r,s)}{2\pi F_{yy}(r,s)}}\, n^{-3/2} r^{-n}.$$

END OF EXAM