1. Let $N = \binom{n}{2}$, the number of pairs of vertices.
   (a) Since there are three choices for each pair of vertices, the answer is 3^N. Alternatively, you could sum the answer for (b): $\sum_{k=0}^{N} \binom{N}{k} 2^k$.
   (b) First construct a simple graph, then orient each of its k edges. Since each edge has two possible orientations, the answer is $\binom{N}{k} 2^k$.

2. If you sketch the wrong graph but get its chromatic polynomial correct, you receive full credit on (b). (But no credit on (a).) You can compute the chromatic polynomial in various ways. Here is one.
   - Color a, giving x.
   - Color b and d differently from a, giving $(x-1)^2$.
   - Color e differently from d, giving $x-1$.
   - Color c differently from a and b, giving $x-2$.

   This gives $P(x) \;=\; x(x-1)^3(x-2)$.
   The number of proper colorings is $P(5)$. If you know this but used the wrong $P(x)$, you still the 2 points.

3. We'll call a sequence without 11100 a free sequence.
   (a) For the first equation, a sequence is either a free sequence or an arbitrary sequence followed by the last 11100 pattern followed by a free sequence.
   To get the second equation, note that every occurrence of 11100 is separated by (possibly empty) free sequences. The resulting summation is a geometric series whose sum is $\frac{F(x)}{1-x^5F(x)}$.
   (b) This is simply a matter of algebra. The answer is $\dfrac{1}{1-2x+x^5}$; that is $P(x)=1$ and $Q(x) = 1-2x+x^5$.
   (c) Multiply both sides by $Q(x)$ and equate coefficients of x^n:

$$f_n - 2f_{n-1} + f_{n-5} \;=\; \begin{cases} 1 & \text{if } n=0, \\ 0 & \text{otherwise.} \end{cases}$$

   Thus $f_n \;=\; 2f_{n-1} - f_{n-5}$ for $n \geq 1$ with $f_0 = 1$ and $f_k = 0$ for $k < 0$.
   This can be derived directly. Here is the idea:

   $(+f_{n-1})$ append a 1 to an $(n-1)$-long free sequence OR
   $(+f_{n-1})$ append a 0 to and $(n-1)$-long free sequence AND
   $(-f_{n-5})$ remove any sequence that ends in 11100.

Remark: This argument can be used for any pattern instead of 11100 provided two copies of the pattern cannot overlap. For example, it does not work with 10110 because 10110110 contains two overlapping copies of 10110.

END OF EXAM