- Please put your name and ID number on your blue book.
- The exam is CLOSED BOOK, but you may have a page of notes.
- Calculators are NOT allowed.
- **You must show your work to receive credit.**

1. (16 pts.) An *oriented* graph is a simple graph in which each edge has been given a direction. In other words, given vertices x and y with $x \neq y$, exactly one of the following is true:
   - There is no edge between x and y.
   - There is an edge from x to y.
   - There is an edge from y to x.

   Obtain formulas for

   (a) the number of oriented graphs with vertex set $\underline{n}$; that is, the number of n-vertex oriented graphs;

   (b) the number of n-vertex oriented graphs having exactly k edges.

2. (a) (2 pts.) Sketch the simple graph $G = (V, E)$ where

$$V = \{a, b, c, d, e\} \qquad E = \big\{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{d, e\}\big\}.$$

   (b) (6 pts.) Compute the chromatic polynomial $P_G(x)$.

   (c) (2 pts.) How many ways can G be properly colored if 5 colors are available?

3. (24 pts.) There are $a_n = 2^n$ n-long sequences of zeroes and ones, including the empty sequence, and so $A(x) = \sum a_n x^n = (1 - 2x)^{-1}$. (You do _not_ need to derive this.) Let f_n be the number of such sequences that do _not_ contain the pattern 11100. Let $F(x) = \sum f_n x^n$.

   (a) Derive either of the two formulas

$$A(x) = F(x) + A(x)x^5 F(x) \qquad A(x) = \sum_{t=0}^{\infty} (F(x)x^5)^t F(x).$$

   (Both formulas are correct. Which you derive will depend on how you think about the problem.)

   (b) Using either of the formulas in (a) and the formula for $A(x)$, find polynomials $P(x)$ and $Q(x)$ so that $F(x) = \frac{P(x)}{Q(x)}$; for example, $F(x)$ might be $\frac{7}{23 - x^9}$.

   (c) Using (b) or otherwise, obtain a simple recursion for f_n for $n \geq 5$. Don't worry about initial conditions.

Final Exam in Center 113

END OF EXAM