1. This can be done in various ways. The simplest may be to choose the pair and then the three other cards as follows:

   pair face value, pair suits, 3 other face values, the 3 suits.

   We get $13 \times \binom{4}{2} \times \binom{12}{3} \times 4^3$.

2. Since $b_1 b_5 = 14$, $b_2 b_4 = 5$ and $b_3 b_3 = 4$, the left child is a 3-leaf tree and the right is also a 3-leaf tree. Since $22 - 14 - 5 = 3 = 1b_3 + 1$, each of these trees has rank 1. In other words, the two 3-leaf trees are the same. By calculations like the previous, each has a 2-leaf left child and a 1-leaf right child.

3. (a) To construct such trees, for each $k \geq 0$ attach $2k$ trees to the root. Since the root contributes one vertex, the Rules of Sum and Product and the sum of a geometric series give

   $$T(x) = \sum_{k=0}^{\infty} x(T(x))^{2k} = \frac{x}{1 - T(x)^2}.$$

   Clearing of fractions and rearranging gives the formula.

   (b) We use the last principle (implicit functions) with $F(x, y) = y^3 - y + x$. Thus $B = -3/2$. Since $F_y = 3y^2 - 1$, we must solve $s^3 - s + r = 0$ and $3s^2 - 1 = 0$. Thus $s = 1/\sqrt{3}$ and $r = 2/3\sqrt{3}$. Hence $C = 3\sqrt{3}\,/2$

4. We use the Principle of Inclusion and Exclusion. (You could use a Venn diagram instead.) You can let $S = \{1, \ldots, 419\}$ or $S = 1, \ldots, 420$ since 420, being a multiple of 2 (and also 5 and 7), will be excluded. Let $S = \{1, \ldots, 420\}$, let S_1 be those which are multiples of 2, S_2 those which are multiples of 5, and S_3 those which are multiples of 7. Thus $S_1 \cap S_3$ are the multiples of 14 and so forth. We have

   $$\begin{array}{ll} |S| = 420|S_1| = 420/2 = 210 & |S_2| = 420/5 = 84 \\ |S_3| = 420/7 = 60 & |S_1 \cap S_2| = 420/(2 \cdot 5) = 42 \\ |S_1 \cap S_3| = 420/(2 \cdot 7) = 30 & |S_2 \cap S_3| = 420/(5 \cdot 7) = 12 \\ |S_1 \cap S_2 \cap S_3| = 420/(2 \cdot 5 \cdot 7) = 6. & \end{array}$$

   Thus the answer is $420 - (210 + 84 + 60) + (42 + 30 + 12) - 6 = 144$. The answer could also have been obtained as $420(1 - 1/2)(1 - 1/5)(1 - 1/7) = 6 \times 4 \times 6 = 144$.

5. This was discussed in class. Number the squares 1 to 16 from left to right, starting with the first row. The cycles are

   $$\begin{array}{rl} 0^\circ: & (1)\,(2)\,(3)\,(4)\,(5)\,(6)\,(7)\,(8)\,(9)\,(10)\,(11)\,(12)\,(13)\,(14)\,(15)\,(16) \\ 90^\circ: & (1, 13, 16, 4)\,(2, 9, 15, 8)\,(3, 5, 14, 12)\,(6, 10, 11, 7) \\ 180^\circ: & (1, 16)\,(2, 15)\,(3, 14)\,(4, 13)\,(5, 12)\,(6, 11)\,(7, 10)\,(8, 9) \\ 270^\circ: & (1, 4, 16, 13)\,(2, 8, 15, 9)\,(3, 12, 14, 5)\,(6, 7, 11, 10) \end{array}$$

   Thus 0° has 16 1-cycles, 180° has 8 2-cycles and 90° and 270° each have 4 4-cycles. How many ways can we choose cycles to get 4 black squares?

   $$0^\circ\colon \binom{16}{4} = 1820, \qquad 90^\circ\colon 4, \qquad 180^\circ\colon \binom{8}{2} = 28, \qquad 270^\circ\colon 4.$$

Thus the answer is $\frac{1}{4}(1820 + 4 + 28 + 4) = 464$.

6. There is no such graph. Suppose G were such a graph. We can construct a spanning tree T for G by removing edges one at a time. Using subscripts to indicate the number of edges, we obtain the sequence $G = G_{25}, G_{24}, G_{23}, G_{22}, G_{21}, G_{20}, G_{19} = T$ since a tree has one less edge than it has vertices. Thus we removed six edges. When an edge e_i is removed from G_i, it must belong to at least one cycle of G_i (since otherwise G_{i-1} would not be connected). Thus, removing e_i destroys at least one cycle of G_i and hence of G. Since we removed six edges, we must have destroyed at least six cycles of G. Thus G must have at least six cycles.

7. By (a), the initial condition can be written either as $F(1) = 1$ or $F(1) = H(1)$. Adding up the moves in (b–d), we have $F(k) + H(n-k) + F(k)$. Since we are told to take the minimum,

$$F(n) = \begin{cases} 1, & \text{if } n = 1, \\ \min\limits_{1 \le k < n} \big(2F(k) + H(n-k)\big), & \text{if } n > 1. \end{cases}$$