- Please put your name and ID number on your blue book.
- The exam is CLOSED BOOK.
- You may have TWO PAGES of notes plus the sheet of principles.
- Calculators are NOT allowed.
- **You must show your work to receive credit.**

1. (12 pts.) How many five-card hands contain one pair, but not two pair or 3 of a kind?

2. (12 pts.) Find the 6-leaf, unlabeled, binary RP-tree whose rank among all such 6-leaf trees is 22.

$$b_1 = b_2 = 1 \quad b_3 = 2 \quad b_4 = 5 \quad b_5 = 14 \quad b_6 = 42$$

3. (24 pts.) Consider unlabled RP-trees in which each vertex has an even number of downward edges (and, of course, one upward edge if it is not the root). I'll draw some pictures on the board. Let t_n be the number of such trees with n vertices and let $T(x)$ be the generating function $\sum t_n x^n$.

   (a) Derive the formula $T(x)^3 - T(x) + x = 0$.

   (b) It can be shown that $t_n \sim An^B C^n$ for some constants A, B and C. Find B and C.

   Note: You need not do (a) in order to be able to do (b).

4. (14 pts.) How many positive integers less than 420 are *not* multiples of 2, 5 or 7? Remember to show how you got your answer. (The list is 1, 3, 9, 11, 13 and so on.)

5. (14 pts.) The squares on a 4×4 board are to be colored red and black so that four squares are red and twelve are black. The only symmetries allowed are rotations of the board by 0°, 90°, 180° and 270°. How many ways can the board be colored?

6. (12 pts.) I claim it is possible to construct a *connected simple graph* that has 20 vertices 25 edges and at most 5 cycles. Nick claims that this is impossible.

   If I am right, construct such a graph. If Nick is right, prove that there is no such graph.

   Note: Different cycles may have edges in common. For example, the simple graph with $V = \{1,2,3,4\}$ and $E = \big\{\{1,2\},\ \{1,3\},\ \{4,2\},\ \{4,3\},\ \{2,3\}\big\}$ has three cycles. The vertices on the cycles are 1,2,3 and 4,2,3 and 1,2,4,3.

THERE ARE MORE PROBLEMS

7. (12 pts.) Suppose the Tower of Hanoi puzzle is modified to have four poles instead of three, say S, E_1, E_2, G. There are n washers on S that must be moved to G with the same restrictions as in the standard Tower of Hanoi (one at a time and no larger on a smaller). We use the following strategy, which may not be the best.

   (a) If $n = 1$, simply move the washer from S to G. If $n > 1$, carry out the following steps.

   (b) Using this 4-pole strategy recursively move the top k washers from S to E_2 for some k with $1 \leq k < n$.

   (c) Using the standard Tower of Hanoi strategy, move the bottom $n-k$ washers from S to G using E_1 as the extra pole. (We can't use E_2 because of washer size.)

   (d) Using this 4-pole strategy recursively move the top k washers from E_2 to G.

   I did not specify the value of k. It is chosen so that the total number of moves is a minimum. Of course, k depends on n.

   Let $H(n)$ be the number of moves for standard Tower of Hanoi and let $F(n)$ be the number of moves for this strategy. Derive a recursion for F, including **INITIAL CONDITIONS**. Your recursion equation should be of the form $F(n) = \min_{1 \leq k < n} (\cdots)$, where $\cdots$ involves F and H. *Don't just state a recursion—derive (i.e., explain) it.*

END OF EXAM