- Please put your name and ID number on your blue book.
- The exam is CLOSED BOOK, but you may have a page of notes.
- Calculators are NOT allowed.
- **You must show your work to receive credit.**

1. (28 pts.) Consider the simple graph $G = (V, E)$ with $V = \{a, b, c, d, e\}$ and $E = \big\{\{a,b\},\ \{a,c\},\ \{b,c\},\ \{a,d\},\ \{a,e\}\big\}$.
   (a) (6 pts.) Sketch the graph G.
   (b) (6 pts.) Give all spanning trees of G.
   (c) (6 pts.) Think of each spanning tree in (b) as rooted at the vertex a. For each of these rooted trees, indicate whether or not it is a lineal spanning tree of G. (A lineal spanning tree is also called a depth first spanning tree.)
   (d) (10 pts.) Compute the chromatic polynomial of G.

The sequences of zeroes and ones that begin and end with zero such that all maximal strings of ones are of odd length are described by the regular expression

$$(00^*(11)^*1)^*\,00^*. \qquad (1)$$

You do not need to prove this. If a_n is the number of such n-long sequences, then the generating function $A(x) = \sum a_n x^n$ has the form

$$\frac{P(x)}{1 - x - 2x^2 + x^3} \quad \text{for some third degree polynomial } P(x).$$

Each of the following problems can be done independently of the others.

2. (10 pts.) Using (1), derive the formula for $A(x)$, including a formula for $P(x)$.
3. (8 pts.) Find k and constants $c_1, c_2, \ldots, c_k$ so that $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ for all sufficiently large n. You need not find the initial conditions.
4. (8 pts.) It turns out that

   $$x^3 - 2x^2 - x + 1 = (x - \alpha)(x - \beta)(x - \gamma)$$

   where $\alpha = -0.801937\ldots$, $\beta = 0.554958\ldots$ and $\gamma = 2.246979\ldots$. Find constants A, B and C so that $a_n \sim A\, n^B\, C^n$. You may express A, B and C in terms of α, β, γ and P, so there is no need for a calculator.
5. (8 pts.) Let $A(x, y)$ be the generating function for the sequences counted by (1), where the coefficient of $x^n y^k$ is the number of n-long sequences with exactly k ones. You do not need to compute this generating function.

   Write down a formula for the average number of ones in an n-long sequence in terms of the coefficients of $A(x, y)$ and related generating functions. An example (but WRONG) of such an expression is $\Big([x^n y^k]\ (A(x,y))^2\Big) \Big/ \Big([x^n]\ A_x(x,1)\Big)^2$, where $A_x = \partial A/\partial x$.

END OF EXAM