1. (30 pts) We want to divide 6 men and 7 women into two teams of 6 people each, plus one referee.

   (a) How many ways can this be done if the teams have no names?

   Ans. We can choose the referee in 13 ways. Look at the first remaining person. We can choose his/her teammates in $\binom{11}{5}$ ways. Thus the answer is $13 \times \binom{11}{5}$.
   Another way: Given an unlabeled choice, there are two ways we can label the teams as 1 and 2. The number of ways to do things with labeled teams is $\binom{13}{6,6,1}$, but, by the preceding sentence, this is twice the desired answer. Hence we have $\frac{1}{2}\binom{13}{6,6,1}$.

   (b) How many ways can this be done if one team is called the "Hackers", the other is called the (Number) "Crunchers", and each team must have 3 women and 3 men?

   Ans. The refree must be a woman and there are 7 ways to choose her. Choose the men for the Hackers and the women for the hackers. Altogether we get $7 \times \binom{6}{3} \times \binom{6}{3}$.

2. (20 pts) Suppose $a_0 = 0$, $a_1 = 1$ and $a_{n+1} = 2a_n - a_{n-1} + 6n$ for $n \geq 1$. Find a simple formula for a_n (not a recursion) and prove it by induction.
   Hint: Compute the first few values of a_n and look for a pattern.

Ans. The pattern is $a_n = n^3$. It is correct for $n = 0$ and $n = 1$. Consider $n \geq 1$. We will prove that $a_{n+1} = (n+1)^3$ by induction. By the recursion and then the induction hypothosis and finally some algebra,

$$a_{n+1} = 2a_n - a_{n-1} + 6n = 2n^3 - (n-1)^3 + 6n = n^3 + 3n^2 + 3n + 1 = (n+1)^3.$$

3. (15 pts) Suppose $n > 1$ Imagine listing the $n!$ permutations of $\{1, 2, \ldots, n\}$ in direct insertion order. What permutation has rank $n!/2$? *Be sure to explain how you got your answer.*

Ans. This is just past the middle of the list of permutations. The first decision in direct insertion order is which of two places to put 2. Since each choice leads to $n!/2$ permutations, we select the second choice and thereafter always select the first choice. This gives us $2, 1, 3, 4, \ldots, n$. In other words, the identity permutation with 1 and 2 switched.

4. (35 pts) We have an unlimited supply of b different kinds of beads. We want to space k of them uniformly around a circle. (In other word, we want a k-long circular list.)

   (a) How many ways can this be done when $k = 6$ and all 6 beads must be different?

   Ans. This is just a circular list without repeats, which was dealt with in Chapter 1. The answer in general is $b(b-1)\cdots(b-k+1)/k$, from which you can work out the answer for $k = 6$.

(b) How many ways can this be done when $k = 6$ and there is no restriction on the beads? *Suggestion*: Use the Burnside Lemma.

Ans. This is an example in the text. For completeness, here's a solution. We can circularly shift the beads by 0, 1, 2, 3, 4, or 5. Here are the results

original	b_1	b_2	b_3	b_4	b_5	b_6
shift by 0	b_1	b_2	b_3	b_4	b_5	b_6
shift by 1	b_2	b_3	b_4	b_5	b_6	b_1
shift by 2	b_3	b_4	b_5	b_6	b_1	b_2
shift by 3	b_4	b_5	b_6	b_1	b_2	b_3
shift by 4	b_5	b_6	b_1	b_2	b_3	b_4
shift by 5	b_6	b_1	b_2	b_3	b_4	b_5

A shift fixes a bead arrangement if and only if the bead shifted into a position is the same as the bead that was there originally. Hence we have the following conditions and resulting counts.

shift	conditions	number
by 0	no conditions	b^6
by 1	$b_1 = b_2 = b_3 = b_4 = b_5 = b_6$	b^1
by 2	$b_1 = b_3 = b_5$ and $b_2 = b_4 = b_6$	b^2
by 3	$b_1 = b_4$, $b_2 = b_5$ and $b_3 = b_6$	b^3
by 4	$b_1 = b_5 = b_3$ and $b_2 = b_6 = b_4$	b^2
by 5	$b_1 = b_6 = b_5 = b_4 = b_3 = b_2$	b^1

Thus the answer is $(b^6 + b + b^2 + b^3 + b^2 + b)/6 = b(b^5 + b^2 + 2b + 2)/6$.

5. (45 pts) Call an $n \times n$ matrix A of zeroes and ones *bad* if there is an index k such that $a_{k,i} = a_{i,k} = 0$ for all i. In other words, the row and column passing through (k, k) consist entirely of zeroes. Let $g(n)$ be the number of $n \times n$ matrices of zeroes and ones which are *not bad*. We want to find $g(n)$ by using inclusion and exclusion.

Let K be a subset of $\{1, 2, \ldots, n\}$. Let $z(K)$ be the number of matrices A that have $a_{i,j} = 0$ whenever either i or j or both belong to K.

(a) Explain why $z(K)$ depends only on $|K|$ and n.

Ans. Various explanations are possible. Here's one. Permute the rows and colums so that those belonging to K now appear as the first $|K|$ rows and first $|K|$ columns.

(b) Let $z_k(n) = z(K)$, where K is any subset of $\{1, 2, \ldots, n\}$ with $|K| = k$. (We can do this because of (a).) Obtain a formula for $z_k(n)$.

Ans. Look at the case when $K = \{1, 2, \ldots, k\}$. The matrix must consist entirely of zeroes except for an $(n-k) \times (n-k)$ block in the lower right which is arbitrary. This block contains $(n-k)^2$ entries and there are 2 choices for each entry. Thus the number of possible matrices is $2^{(n-k)^2}$.

(c) Using inclusion and exclusion, express $g(n)$ in terms of the $z_k(n)$. (Recall that $g(n)$ is the number of $n \times n$ (0,1)-matrices which are not bad.)

Ans. In the notation in the Principle of Inclusion and Exclusion in the text, Let S_i be the set of matrices in which row i and column i consist entirely of zeroes. Since N_r is the sum of $z(K)$ over all K with $|K| = r$ and since there are $\binom{n}{r}$ such K's, we have $N_r = \binom{n}{r} z_r(n)$. Thus the number of matrices which are not bad is $\sum_{i=0}^{n} (-1)^i \binom{n}{i} z_i(n)$.

6. (35 pts) Let t_n be the number of n-vertex RP-trees in which *no vertex has exactly one child* and let $T(x) = \sum_{n=1}^{\infty} t_n x^n$ be the associated generating function.

(a) Derive a simple equation for $T(x)$. You **MUST** explain how you derived the equation, but you do *not* need to solve the equation for $T(x)$.

Ans. You can form such a tree by taking a vertex (as a root) and taking a list of k trees to join to it, with all values of k allowed except $k = 1$. Thus

$$T(x) = x \sum_{\substack{k \geq 0 \\ k \neq 1}} T(x)^k = x \left(\sum_{k=0}^{\infty} T(x)^k - T(x)^1 \right) = x \left(\frac{1}{1 - T(x)} - T(x) \right).$$

(b) If the equation you derived in (a) is correct, it can be solved to obtain

$$T(x) = \frac{1 - \sqrt{(1-3x)/(1+x)}}{2}. \qquad \text{(You can take this as given.)}$$

There are numbers A, B, and C such that $t_n \sim A n^B C^n$. Obtain a simple explicit value for C. (You do not need to determine A or B.)
Hint: Use the ideas from Section 12.4.

Ans. One could use Principle 12.5; however, it is simpler to realize that the exponential part is r^{-n}, where r is the radius of convergence. In this case, the radius of convergence r is due to the square root being zero or the expression inside the square root becoming infinite. The first happens when $x = 1/3$ and the second when $x = -1$. Thus $r = 1/3$ and $C = 3$.

7. (20 pts) The analysis of the running time of a certain algorithm have n inputs led to the following summation. Estimate $T(n)$

$$T(n) = \sum_{k=2}^{n} \frac{2^k}{\ln k}.$$

Ans. Call the terms a_k. The terms are increasing and, for large k,

$$\frac{a_k}{a_{k+1}} = \frac{2^k}{\ln k} \frac{\ln(k+1)}{2^{k+1}} = \frac{\ln(k+1)}{2 \ln k} \sim 1/2.$$

To use Principle 12.3, we need to look at the sum as starting at $k = n$ and going to $k = 0$. We find that

$$T(n) \sim a_n \frac{1}{1 - 1/2} = 2a_n = \frac{2^{n+1}}{\ln n}.$$