- Please put your name and ID number on your exam. If you are not using a blue book, put your name on every page.
- The exam is CLOSED BOOK, but ONE SHEET of notes is allowed.
- **You must show your work to receive credit.**

1. (30 pts) We want to divide 6 men and 7 women into two teams of 6 people each, plus one referee.

   (a) How many ways can this be done if the teams have no names?

   (b) How many ways can this be done if one team is called the "Hackers", the other is called the (Number) "Crunchers", and each team must have 3 women and 3 men?

2. (20 pts) Suppose $a_0 = 0$, $a_1 = 1$ and $a_{n+1} = 2a_n - a_{n-1} + 6n$ for $n \geq 1$. Find a simple formula for a_n (not a recursion) and prove it by induction.
   Hint: Compute the first few values of a_n and look for a pattern.

3. (15 pts) Suppose $n > 1$ Imagine listing the $n!$ permutations of $\{1, 2, \ldots, n\}$ in direct insertion order. What permutation has rank $n!/2$? *Be sure to explain how you got your answer.*

4. (35 pts) We have an unlimited supply of b different kinds of beads. We want to space k of them uniformly around a circle. (In other word, we want a k-long circular list.)

   (a) How many ways can this be done when $k = 6$ and all 6 beads must be different?

   (b) How many ways can this be done when $k = 6$ and there is no restriction on the beads? *Suggestion*: Use the Burnside Lemma.

5. (45 pts) Call an $n \times n$ matrix A of zeroes and ones *bad* if there is an index k such that $a_{k,i} = a_{i,k} = 0$ for all i. In other words, the row and column passing through (k,k) consist entirely of zeroes. Let $g(n)$ be the number of $n \times n$ matrices of zeroes and ones which are *not bad*. We want to find $g(n)$ by using inclusion and exclusion.

   Let K be a subset of $\{1,2,\ldots,n\}$. Let $z(K)$ be the number of matrices A that have $a_{i,j} = 0$ whenever either i or j or both belong to K.

   (a) Explain why $z(K)$ depends only on $|K|$ and n.

   (b) Let $z_k(n) = z(K)$, where K is any subset of $\{1,2,\ldots,n\}$ with $|K| = k$. (We can do this because of (a).) Obtain a formula for $z_k(n)$.

   (c) Using inclusion and exclusion, express $g(n)$ in terms of the $z_k(n)$. (Recall that $g(n)$ is the number of $n \times n$ (0,1)-matrices which are not bad.)

6. (35 pts) Let t_n be the number of n-vertex RP-trees in which *no vertex has exactly one child* and let $T(x) = \sum_{n=1}^{\infty} t_n x^n$ be the associated generating function.

   (a) Derive a simple equation for $T(x)$. You **MUST** explain how you derived the equation, but you do *not* need to solve the equation for $T(x)$.

   (b) If the equation you derived in (a) is correct, it can be solved to obtain

   $$T(x) = \frac{1 - \sqrt{(1-3x)/(1+x)}}{2}. \qquad \text{(You can take this as given.)}$$

   There are numbers A, B, and C such that $t_n \sim An^B C^n$. Obtain a simple explicit value for C. (You do not need to determine A or B.)
   Hint: Use the ideas from Section 12.4.

7. (20 pts) The analysis of the running time of a certain algorithm have n inputs led to the following summation. Estimate $T(n)$

$$T(n) = \sum_{k=2}^{n} \frac{2^k}{\ln k}.$$

END OF EXAM