1. (45 pts.) (a) Compute the rank of the permutation $1,5,3,4,2$ when the permutations of $\{1,2,3,4,5\}$ are listed in direct insertion order.

**Ans.** If we start numbering positions with 0 and let $p(i)$ be the position where $i$ is inserted, then the rank is $(5!/2!)\,p(2) + (5!/3!)\,p(3) + (5!/4!)\,p(4) + (5!/5!)\,p(5)$. Since $p(2)=0$, $p(3)=1$, $p(4)=1$, and $p(5)=3$, the rank is 28. You could also draw the relevant part of the decision tree here and in (b) and (c) to figure things out.

(b) What permutation immediately follows $1,5,3,4,2$ in direct insertion order?

**Ans.** Increase $p(5)$ by 1, giving $5,1,3,4,2$.

(c) What permutation immediately follows $1,5,3,4,2$ in lex order?

**Ans.** Starting from the right, we look for the first position that can be increased. The 2 can't be changed since there's nothing to replace it with. The 4 can only be replaced by 2, which is a decrease. The 3 can be replaced by 2 or 4. To increase, use 4. Next arrange the left over 2 and 3 to give the leftmost possibility. We have $1,5,4,2,3$.

2. (20 pts.) Find the unlabeled rooted plane tree with 7 leaves and rank 60.

**Ans.** We have $56 = b_1b_6 + b_2b_5 \le 60 < b_1b_6 + b_2b_5 + b_3b_4$. Also $60-56 = 0b_4+4$. Thus the left subtree of the root has 3 leaves and rank 0 while the right has 4 leaves and rank 4. Each of these is easy to determine: Rank 0 means keep all the branching to the right. Rank $b_k - 1$ means keep all the branching to the left.

$$b_1 = b_2 = 1, \quad b_3 = 2, \quad b_4 = 5, \quad b_5 = 14, \quad b_6 = 42, \quad b_7 = 132.$$

3. (30 pts.) Let $C_n(k)$ be the number of times position $k$ is changed in the Gray code for $n$-long vectors of zeroes and ones given in the text. You may use the following fact without proving it:

$$C_n(k) = \begin{cases} 1 & \text{if } k = 1, \\ 2C_{n-1}(k-1) & \text{if } 1 < k \le n, \\ 0 & \text{if } k > n. \end{cases}$$

(a) Tabulate values of $C_n(k)$ for $1 \le k \le n \le 4$.

**Ans.** This is straightforward.

(b) State and prove a simple formula for $C_n(k)$ that does not involve a recursion.

**Ans.** You can guess from (a) that $C_n(k) = 2^{k-1}$ for $1 \le k \le n$ and zero otherwise. This can proved by induction. Should we induct on $n$ or $k$? Either will work However, since the answer does not depend on $n$ and "$C_n(k) = 1$ if $k = 1$" looks like an initial condition, it may be somewhat easier to induct on $k$. (It is.) As noted, this is the base case. For $k$ in the range $2 \le k \le n$, we have

$$\begin{aligned} C_n(k) &= 2C_{n-1}(k-1) && \text{by the given formula,} \\ &= 2 \times 2^{k-2} && \text{by } \mathcal{A}(k-1), \\ &= 2^{k-1}. \end{aligned}$$

4. (30 pts.) Let $b_n$ be the number of binary unlabeled rooted plane trees with $n$ leaves. It is known that $b_n < 4b_{n-1}$ for $n > 1$ and you may use this fact without proof. (It can be proved by using Exercise 9.1.12.)

   (a) Show that, for more than $b_n/4$ of these trees with $n$ leaves, the left subtree consists of just a single leaf when $n > 1$.
   In other words, show that if the two edges leading from the root go to trees $T_1$ and $T_2$, then we have $|T_1| = 1$ in more than $b_n/4$ of the cases.

   *Hint*: What does the term $b_i b_{n-i}$ in $b_n = b_1 b_{n-1} + b_2 b_{n-2} + \cdots + b_{n-1} b_1$ count?

**Ans.** Let $|T|$ be the number of *leaves* of $T$ Following the hint, $b_i b_{n-i}$ counts those $T$ with $|T| = n$, $|T_1| = i$ and $|T_2| = n - i$. Thus the number with $|T_1| = 1$ is $b_1 b_{n-1} = b_{n-1} > b_n/4$, where the inequality follows from $B_n < 4b_{n-1}$.

   (b) Find a constant $P > 1/4$ such that the following statement is true for $n > 1$ about those binary trees with $n$ leaves.

   "In more than $Pb_n$ of them, the left subtree contains at most 2 leaves."

**Ans.** From $b_N < 4b_{N-1}$ at $N = n$ and $N = n - 1$, we have $b_n < 4b_{n-1} < 4^2 b_{n-2}$. Thus $b_{n-2} > b_n/16$. Reasoning as in (a), the number of such trees is greater than

$$b_1 b_{n-1} + b_2 b_{n-2} = b_{n-1} + b_{n-2} < b_n/4 + b_n/16 = 4b_n/16.$$

Thus $P$ can have any value such that $1/4 < P \leq 5/16$.