- Please put your name and ID number on your exam. If you are not using a blue book, put your name on every page.
- The exam is CLOSED BOOK, but ONE SHEET of notes is allowed.
- **You must show your work to receive credit.**

1. (45 pts.) (a) Compute the rank of the permutation $1, 5, 3, 4, 2$ when the permutations of $\{1, 2, 3, 4, 5\}$ are listed in direct insertion order.
   (b) What permutation immediately follows $1, 5, 3, 4, 2$ in direct insertion order?
   (c) What permutation immediately follows $1, 5, 3, 4, 2$ in lex order?

2. (20 pts.) Find the unlabeled rooted plane tree with 7 leaves and rank 60.

$$b_1 = b_2 = 1, \quad b_3 = 2, \quad b_4 = 5, \quad b_5 = 14, \quad b_6 = 42, \quad b_7 = 132.$$

3. (30 pts.) Let $C_n(k)$ be the number of times position k is changed in the Gray code for n-long vectors of zeroes and ones given in the text. You may use the following fact without proving it:

$$C_n(k) = \begin{cases} 1 & \text{if } k = 1, \\ 2C_{n-1}(k-1) & \text{if } 1 < k \leq n, \\ 0 & \text{if } k > n. \end{cases}$$

   (a) Tabulate values of $C_n(k)$ for $1 \leq k \leq n \leq 4$.
   (b) State and prove a simple formula for $C_n(k)$ that does not involve a recursion.

4. (30 pts.) Let b_n be the number of binary unlabeled rooted plane trees with n leaves. It is known that $b_n < 4b_{n-1}$ for $n > 1$ and you may use this fact without proof. (It can be proved by using Exercise 9.1.12.)
   (a) Show that, for more than $b_n/4$ of these trees with n leaves, the left subtree consists of just a single leaf when $n > 1$.
   In other words, show that if the two edges leading from the root go to trees T_1 and T_2, then we have $|T_1| = 1$ in more than $b_n/4$ of the cases.

   Hint: What does the term $b_i b_{n-i}$ in $b_n = b_1 b_{n-1} + b_2 b_{n-2} + \cdots + b_{n-1} b_1$ count?

   (b) Find a constant $P > 1/4$ such that the following statement is true for $n > 1$ about those binary trees with n leaves.

   "In more than Pb_n of them, the left subtree contains at most 2 leaves."

To receive credit, you must prove that your value for P works.

END OF EXAM