1. For each of the following,
   EITHER give an example of the thing that is described
   OR explain why none exists.

   (a) A surjection from $\{1,2,3\}$ to $\{a,b,c,d\}$.

   A. Impossible since a function from a 3-set to a 4-set cannot hit everything in the 4-set.

   (b) An injection from $\{1,2,3\}$ to $\{a,b,c,d\}$.

   A. There are 24 examples. One is $f(1)=a$, $f(2)=b$, $f(3)=c$.

   (c) A permutation f of $\{1,2,3,4\}$ such that f^{12} is NOT the identity function. The identity function is the function g such that $g(x)=x$ for all x.
   Remember that $f^{12}(x)$ is $f(f(\cdots f(x)))$, not $(f(x))^{12}$.

   A. Impossible. The cycles of a permutation on a n-set all have lengths at most n. Hence f^k must be the identity if k is divisible by all numbers from 2 to n. When $n=4$, this tells us that it suffices to have k a multiple of 12. (You only need to give an argument like this for $n=4$.)

   (d) An involution f of $\{1,2,3,4,5,6\}$ with exactly 4 cycles.

   A. There are 45 examples, all of which have two 1-cycles and two 2-cycles. One is $(1)(2)(3,4)(5,6)$. As mentioned in class, it is permissible to omit the 1-cycles, so this could be written $(3,4)(5,6)$.

2. How many 6 card hands contain 3 pairs?
   Important: Give a careful justification of how you found your answer, not just a bunch of numbers multiplied together without explanation.

A. See Ex. 1.3.1. Here's one solution. Choose the 3 face values for the pairs and then choose the suits for each of the face values:

$$\binom{13}{3}\times\binom{4}{2}\times\binom{4}{2}\times\binom{4}{2}=\frac{13\times 12\times 11\times 6^3}{6}=61,776.$$

3. A magazine article lists 8 yes/no questions. Each question must be answered "yes," "no" or "don't know". The article says that a person is credulous if he or she answers at least 6 of the questions with a "yes". How many ways are there to answer the questions so that you receive the label "credulous?"

A. Let $A(k)$ be the number of ways to answer with exactly k "yes" answers. The solution is $A(6)+A(7)+A(8)$. To compute $A(k)$, choose k questions to answer "yes" and give non-yes answers to the remaining $8-k$. This can be done in $\binom{8}{k}2^{8-k}$ ways since each of the $8-k$ questions has 2 possible answers. We obtain

$$A(6)=\binom{8}{6}2^2=112 \quad A(7)=\binom{8}{7}2^1=16 \quad A(8)=1.$$

Thus the answer is 129.

4. Let S be an n-set. It was shown in the text (and in class) that the number of subsets that can be formed from S is 2^n.

   Generalize this: State and prove a formula for the number of multisets that can be formed from S where *each element is repeated at most k times.*
   When $k = 1$, your formula should become 2^n.

A. Each element can be used from 0 to k times in the multiset, thus there are $k+1$ choices for how many times to use each element. We make a choice for the number of times to use the first element of S, and then for the second, and then for the third, and so on. The Rule of Product gives $(k+1)^n$.

5. Let $S(n,k)$ be the Stirling numbers of the second kind; that is, $S(n,k)$ is the number of ways to partition an n-set into k unordered blocks. Prove that

$$S(n,k) = \sum_{j=1}^{n} \binom{n-1}{j-1} S(n-j, k-1) \quad \text{for } n > 0 \text{ and } k > 0.$$

A. See Ex. 1.4.6(c). We can let the set be $\{1, 2, \ldots, n\}$. Look at the block containing n. It can contain any number of additional elements. The remaining elements must be divided into $k-1$ blocks. Let j be the number of elements in the block containing n. By the above discussion, this contributes $\binom{n-1}{j-1} S(n-j, k-1)$ different partitions to the count.